Symmetric composition algebras and exceptional simple Lie algebras

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Composition algebras

Symmetric composition algebras

Triality

Freudenthal-Tits Magic Square

Composition algebras

Definition

A *composition algebra* over a field k is a triple $(C, \cdot, n)$ where

- C is a vector space over k,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to k$ is a *multiplicative* regular quadratic form:
  - $n(x \cdot y) = n(x)n(y)\ \forall x, y \in C$,
  - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate (if char $k = 2$ we also allow the radical of the polar form to be non isotropic and of dimension 1).

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The unital composition algebras are termed *Hurwitz algebras*.

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

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They are endowed with an antiautomorphism, the *standard conjugation*:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

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Moreover,

$$n(xy, z) = n(y, \bar{x}z) = n(x, z\bar{y}),$$

for any x, y, z. (The adjoint of the left (right) multiplication by x is the left (right) multiplication by $\bar{x}$.)

Cayley-Dickson doubling process

Let B be an associative Hurwitz algebra with norm n such that the polar form is nondegenerate, and let λ be a nonzero scalar in the ground field k. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and regular quadratic form that extend those on B:

$$(a + bu)(c + du) = (ac + \lambda \bar{d} b) + (da + b\bar{c})u,$$
$$n(a + bu) = n(a) - \lambda n(b).$$

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Notation: $CD(A, \mu, \lambda) := CD\big(CD(A, \mu), \lambda\big).$

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field k is isomorphic to one of the following:

(i) *The ground field k.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = k1 + kv$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

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In particular, any Hurwitz algebra is finite-dimensional.

General composition algebras

Corollary

The dimension of any finite-dimensional composition algebra is restricted to 1, 2, 4 *or* 8.

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Proof.

- Take any element a of C with $n(a) \neq 0$. Then the norm of $e = \frac{1}{n(a)} a^2$ is 1.
- Consider the new multiplication on C (Kaplansky's trick):

$$x \cdot y = (R_e^{-1} x)(L_e^{-1} y).$$

- Then $(C, \cdot, n)$ is a Hurwitz algebra with unity $1 = e^2$.

□

General composition algebras

Theorem (E.–Pérez-Izquierdo 97)

There are infinite-dimensional composition algebras of arbitrary infinite dimension, even with a one-sided unity!

The split Hurwitz algebras

There are, up to isomorphism, four 'split' (i.e., either $\dim C = 1$ or $\exists x$ s.t. $n(x) = 0$) Hurwitz algebras:

$$k, \qquad k \times k, \qquad \mathrm{Mat}_2(k), \qquad C(k).$$

The split Cayley algebra

Canonical basis of the *split Cayley algebra* $C(k) = CD\big(\mathrm{Mat}_2(k), -1\big)$:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

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$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

$n(e_1, e_2) = n(u_i, v_i) = 1, \quad$ (otherwise 0)

$e_1^2 = e_1, \quad e_2^2 = e_2,$

$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$

$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$

$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2},\ v_i v_{i+1} = -v_{i+1} v_i = u_{i+2},$ (indices modulo 3)

otherwise 0.

Composition algebras

Symmetric composition algebras

Triality

Freudenthal-Tits Magic Square

Okubo, 1978

Assume the ground field k contains a cubic primitive root ω of 1 (in particular, char $k \neq 3$).
On the vector space $S = \mathfrak{sl}_3(k)$ of zero trace 3×3 matrices over k, consider the 'new' multiplication:

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1.$$

Okubo, 1978

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$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1.$$

$(S, *, n)$ is a composition algebra, with $n(x) = -\frac{1}{2} \operatorname{tr}(x^2)$
(valid in characteristic 2!)

Okubo, 1978

This composition algebra is not unital, but satisfies a nice property:

$$n(x * y, z) = n(x, y * z)$$

for any x, y, z.
(Associativity of the norm: the adjoint of the left multiplication by x is the right multiplication by x.)

Symmetric composition algebras

Definition

A composition algebra $(S, *, n)$ is said to be *symmetric* if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in S$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in S$.

Examples

Let C be a Hurwitz algebra with norm n.

- *Para-Hurwitz algebras (Okubo-Myung 1980)*: Consider the new multiplication on C:

  $$x \bullet y = \bar{x} \cdot \bar{y}.$$

  Then $(C, \bullet, n)$ is a composition algebra, which will be denoted by $\bar{C}$ for short.

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  The unity of C becomes a *para-unit* in $\bar{C}$, that is, an element e such that $e \bullet x = x \bullet e = n(e, x)e - x$. If the dimension is at least 4, the para-unit is unique, and it is the unique idempotent that spans the commutative center of the para-Hurwitz algebra.

Examples

- *Petersson algebras (1969)*: Let τ be an automorphism of C with $\tau^3 = 1$, and consider the new multiplication defined on C by means of:
$$x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y}).$$
The algebra $(C, *, n)$ is a symmetric composition algebra, which will be denoted by $\bar{C}_\tau$ for short.

Okubo algebras

Let $\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ be a canonical basis of $C(k)$. Then the linear map $\tau_{st} : C(k) \to C(k)$ determined by the conditions:

$$\tau_{st}(e_i) = e_i,\ i = 1, 2;\ \tau_{st}(u_i) = u_{i+1},\ \tau_{st}(v_i) = v_{i+1} \text{ (indices modulo 3)},$$

is clearly an order 3 automorphism of $C(k)$.

Definition

The associated Petersson algebra $P_8(k) = \overline{C(k)}_{\tau_{st}}$ is called the *pseudo-octonion algebra* over the field k. It is isomorphic to the algebra originally defined by Okubo.

The forms of $P_8(k)$ are called *Okubo algebras* [E.-Myung 1990].

Classification

Theorem (Okubo-Osborn 81, E.–Myung 91,93, E.–Pérez-Izquierdo 96, E. 97)

Any symmetric composition algebra is either:

- *a para-Hurwitz algebra,*
- *a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),*
- *an Okubo algebra.*

Classification

Moreover:

- If char $k \neq 3$ and $\exists \omega \neq 1 = \omega^3$ in k, then any Okubo algebra is, up to isomorphism, the algebra A_0 of zero trace elements in a central simple degree 3 associative algebra with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$

and norm $n(x) = -\frac{1}{2}\,\mathrm{tr}(x^2)$.

Classification

- If char $k \neq 3$ and $\nexists \omega \neq 1 = \omega^3$ in k, then any Okubo algebra is, up to isomorphism, the algebra $S(A, j)_0 = \{x \in A_0 : j(x) = -x\}$, where (A, j) is a central simple degree three associative algebra over $k[\omega]$ and j is a $k[\omega]/k$-involution of second kind, with multiplication and norm as above.

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Example

$\mathbb{C} = \mathbb{R}[\omega]$, $\mathfrak{su}_3 = \{x \in \mathrm{Mat}_3(\mathbb{C}) : \mathrm{tr}(x) = 0,\ \bar{x}^t = -x\}$.
The corresponding Okubo algebra is a *division algebra*.

Classification

If char $k \neq 3$ and k contains the cubic roots of 1, for $0 \neq \alpha, \beta \in k$, $A = \text{alg}\langle x, y : x^3 = \alpha,\ y^3 = \beta,\ xy = \omega yx \rangle$ is a central simple degree 3 associative algebra.

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Take the elements $x_{ij} = \frac{\omega^{ij}}{\omega^2 - \omega} x^i y^j$, so that

$$A_0 = \mathrm{span}\left\{x_{ij} : -1 \leq i, j \leq 1,\ (i,j) \neq (0,0)\right\}.$$

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The multiplication table of the corresponding Okubo algebra is the following:

Classification

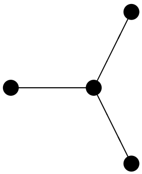
$*$	$x_{1,0}$	$x_{-1,0}$	$x_{0,1}$	$x_{0,-1}$	$x_{1,1}$	$x_{-1,-1}$	$x_{-1,1}$	$x_{1,-1}$
$x_{1,0}$	$-\alpha x_{-1,0}$	0	0	$x_{1,-1}$	0	$x_{0,-1}$	0	$\alpha x_{-1,-1}$
$x_{-1,0}$	0	$-\alpha^{-1}x_{1,0}$	$x_{-1,1}$	0	$x_{0,1}$	0	$\alpha^{-1}x_{1,1}$	0
$x_{0,1}$	$x_{1,1}$	0	$-\beta x_{0,-1}$	0	$\beta x_{1,-1}$	0	0	$x_{1,0}$
$x_{0,-1}$	0	$x_{-1,-1}$	0	$-\beta^{-1}x_{0,1}$	0	$\beta^{-1}x_{-1,1}$	$x_{-1,0}$	0
$x_{1,1}$	$\alpha x_{-1,1}$	0	0	$x_{1,0}$	$-\alpha\beta x_{-1,-1}$	0	$\beta x_{0,-1}$	0
$x_{-1,-1}$	0	$\alpha^{-1}x_{1,-1}$	$x_{-1,0}$	0	0	$-(\alpha\beta)^{-1}x_{1,1}$	0	$\beta^{-1}x_{0,1}$
$x_{-1,1}$	$x_{0,1}$	0	$\beta x_{-1,-1}$	0	0	$\alpha^{-1}x_{1,0}$	$-\alpha^{-1}\beta x_{1,-1}$	0
$x_{1,-1}$	0	$x_{0,-1}$	0	$\beta^{-1}x_{1,1}$	$\alpha x_{-1,0}$	0	0	$-\alpha\beta^{-1}x_{-1,1}$

Classification

This multiplication table is valid in characteristic 3(!!), and the Okubo algebras with this multiplication table exhaust:

- The Okubo algebras over fields of characteristic 3.
- The Okubo algebras with isotropic norm over arbitrary fields.

Triality



The simple Lie algebra of type D_4 contains outer automorphisms of order 3.

Symmetric composition algebras and triality

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$$l_x r_x = n(x)\mathrm{id} = r_x l_x \quad \Longrightarrow \quad \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix}^2 = n(x)\mathrm{id}$$

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Therefore, the map $x \mapsto \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix}$ extends to an isomorphism of algebras with involution

$$\Phi : \big(\mathfrak{Cl}(S, n), \tau\big) \longrightarrow \big(\mathrm{End}(S \oplus S), \sigma_{n \perp n}\big)$$

Spin group

Consider the *spin group*:

$$\mathrm{Spin}(S, n) = \{u \in \mathfrak{Cl}(S, n)_{\bar{0}}^{\times} : u \cdot x \cdot u^{-1} \in S,\ u \cdot \tau(u) = 1,\ \forall x \in S\}.$$

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For any $u \in \mathrm{Spin}(S,n)$,

$$\Phi(u) = \begin{pmatrix} \rho_u^- & 0 \\ 0 & \rho_u^+ \end{pmatrix}$$

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For any $u \in \mathrm{Spin}(S,n)$,

$$\Phi(u) = \begin{pmatrix} \rho_u^- & 0 \\ 0 & \rho_u^+ \end{pmatrix}$$

for some $\rho_u^{\pm} \in O(S,n)$ such that

$$\chi_u(x * y) = \rho_u^+(x) * \rho_u^-(y)$$

for any $x, y \in S$, where $\chi_u(x) = u \cdot x \cdot u^{-1}$.

Spin group

This last condition is equivalent to:

$$\langle \chi_u(x), \rho_u^+(y), \rho_u^-(z) \rangle = \langle x, y, z \rangle$$

for any $x, y, z \in S$, where

$$\langle x, y, z \rangle = n(x, y * z),$$

and this has cyclic symmetry!!

$$\langle x, y, z \rangle = \langle y, z, x \rangle.$$

Spin group

Theorem

Let $(S, *, n)$ be an eight-dimensional symmetric composition algebra. Then:

$$\mathrm{Spin}(S, n) \simeq \{(f_0, f_1, f_2) \in SO(S, n)^3 : \\ f_0(x * y) = f_1(x) * f_2(y) \ \forall x, y \in S\}.$$

Moreover, the set of related triples *(the set on the right hand side) has cyclic symmetry.*

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Moreover, the set of related triples *(the set on the right hand side) has cyclic symmetry.*

The cyclic symmetry on the right hand side induces an outer automorphism of order 3 (*triality automorphism*) of $\mathrm{Spin}(S, n)$.

The Principle of Triality

Theorem

Let $(S, *, n)$ be an eight-dimensional symmetric composition algebra. Then, for any $f_0 \in SO(S, n)$, there are elements $f_1, f_2 \in SO(S, n)$, unique up to scalar multiplication of both by -1, such that (f_0, f_1, f_2) is a related triple.

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Remark

Any of the projections $\pi_i : \mathrm{Spin}(S, n) \to SO(S, n)$, $(f_0, f_1, f_2) \mapsto f_i$ gives a *double cover* of $SO(S, n)$.

Local version: Principle of Local Triality

Theorem

Let $(S, *, n)$ *be an eight-dimensional symmetric composition algebra. Then, for any* $d_0 \in \mathfrak{so}(S, n)$*, there are unique elements* $d_1, d_2 \in \mathfrak{so}(S, n)$ *such that*

$$d_0(x * y) = d_1(x) * y + x * d_2(y),$$

for any $x, y \in S$*.*

Triality Lie algebra

Definition

For any symmetric composition algebra $(S, *, n)$, the Lie algebra

$$\mathfrak{tri}(S,*,n) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S,n)^3 : \\ d_0(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y \in S\}$$

is called the *triality Lie algebra* of $(S, *, n)$.

Triality Lie algebra

Proposition

- *The map* $\theta : \mathfrak{tri}(S,*,n) \to \mathfrak{tri}(S,*.n)$, $(d_0,d_1,d_2) \mapsto (d_1,d_2,d_0)$, *is a Lie algebra automorphism.*
- *If* $\dim S = 8$, *any of the projections* $\mathfrak{tri}(S,*,n) \to \mathfrak{so}(S,n)$, $(d_0,d_1,d_2) \mapsto d_i$, *is an isomorphism of Lie algebras.*
- *If* $\mathrm{char}\, k \neq 2$, *for any* $x, y \in S$, *consider the triple:*

$$t_{x,y} = \left(\sigma_{x,y}, \frac{1}{2}n(x,y)id - r_x l_y, \frac{1}{2}n(x,y)id - l_x r_y\right),$$

where $\sigma_{x,y} : z \mapsto n(x,z)y - n(y,z)x$. *Then*

$$\mathfrak{tri}(S,*,n) = \sum_{i=0}^{2} \theta^i(t_{S,S}),$$

$$[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}.$$

Composition algebras

Symmetric composition algebras

Triality

Freudenthal-Tits Magic Square

A construction of Lie algebras from symmetric composition algebras

Let $(S, *, n)$ and $(S', \star, n')$ be two symmetric composition algebras over a field k of characteristic $\neq 2$. One can construct a Lie algebra as follows:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(S \otimes S')\Big),$$

with bracket given by:

A construction of Lie algebras from symmetric composition algebras

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d'_i(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = n'(x', y')\theta^i(t_{x,y}) + n(x, y)\theta'^i(t'_{x',y'})$,

Freudenthal-Tits Magic Square

		dim S'			
$\mathfrak{g}(S, S')$		1	2	4	8
	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
dim S	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Some notes on Freudenthal-Tits Magic Square

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- Tits construction of the Magic Square involves a Hurwitz algebra and a simple Jordan algebra of degree 3.
- None of these constructions explain the symmetry of the Magic Square.
- Tits construction is equivalent, in a natural way, to the above construction using para-Hurwitz algebras.

That's all. Thanks