Symmetry of the Tits construction of the exceptional simple Lie algebras

Alberto Elduque
(joint work with Susumu Okubo)

Universidad de Zaragoza

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1. Tits construction

2. A more symmetric construction

3. Lie algebras with S_4-symmetry

4. S_4-symmetry in Tits construction

Tits construction

(For simplicity assume $k = \mathbb{C}$ throughout)

- C a unital composition algebra:
  - $n : C \to k$ nondegenerate quadratic form, $n(ab) = n(a)n(b)$ $\forall a, b$.
  - $a^2 - t(a)a + n(a)1 = 0$ $\forall a \in C$ $(t(a) = n(a, 1) = n(a+1) - n(a) - 1)$.
  - $C_0 = \{a \in C : t(a) = 0\}$, $\bar{a} = t(a)1 - a$ is the canonical involution,
  - $\forall a, b \in C$, $D_{a,b} : c \mapsto [[a, b], c] + 3(a, b, c)$ is a derivation.
- J a degree three simple Jordan algebra:
  - $J = H_3(C', *)$, $(x_{ij})^* = (\bar{x}_{ji})$.
  - $tr : J \to k$ usual trace, $J_0 = \{x \in J : tr(x) = 0\}$.
  - $\forall x, y \in J$, $d_{x,y} : z \mapsto x(yz) - y(xz) = -(x, z, y)$ is a derivation.

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Then [Tits, 1966],

$$\mathcal{T}(C,J)=\mathfrak{der}\,C\oplus(C_0\otimes J_0)\oplus\mathfrak{der}\,J$$

is a Lie algebra under the Lie bracket:

- $\mathfrak{der}\,C$ and $\mathfrak{der}\,J$ are Lie subalgebras and $[\mathfrak{der}\,C,\mathfrak{der}\,J]=0$,
- $[D,a\otimes x]=D(a)\otimes x$,
- $[d,a\otimes x]=a\otimes d(x)$,
- $[a\otimes x,b\otimes y]=\frac{1}{3}tr(xy)D_{a,b}+\Big([a,b]\otimes\big(xy-\frac{1}{3}tr(xy)1\big)\Big)+2t(ab)d_{x,y}$,

$\forall a,b\in C_0$, $\forall x,y\in J_0$, $\forall D\in\mathfrak{der}\,C$, $\forall d\in\mathfrak{der}\,J$.

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Freudenthal Magic Square

$C \backslash C'$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

The square is symmetric on C, C', even though these two algebras play very different roles.

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$$\mathfrak{g}(C,C') = \mathfrak{tri}(C) \oplus \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right)$$

where

$$\mathfrak{tri}(C) = \{(d_0,d_1,d_2) \in \mathfrak{so}(C,n) : \overline{d_0(\overline{ab})} = d_1(a)b + ad_2(b)\ \forall a,b \in C\}$$

is the *triality Lie algebra*, and with Lie bracket given by

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$$\mathfrak{tri}(C) = \{(d_0, d_1, d_2) \in \mathfrak{so}(C, n) : \overline{d_0(\overline{ab})} = d_1(a)b + ad_2(b) \ \forall a, b \in C\}$$

is the *triality Lie algebra*, and with Lie bracket given by

- $\mathfrak{tri}(C)$ and $\mathfrak{tri}(C')$ are subalgebras with $[\mathfrak{tri}(C), \mathfrak{tri}(C')] = 0$,
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$t_{a,b} = \Big(n(a, .)b - n(b, .)a, \tfrac{1}{2}(L_{\bar{b}}L_a - L_{\bar{a}}L_b), \tfrac{1}{2}(R_{\bar{b}}R_a - R_{\bar{a}}R_b)\Big)$,

$\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$.

A more symmetric construction (cont.)

Let C and C' be unital composition algebras, consider the vector space

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Lie algebras with S_4-symmetry

$$S_4 = \langle \tau_1 = (12)(34), \tau_2 = (23)(14), \varphi = (123), \tau = (12) \rangle.$$

Let $\mathfrak{g}$ be a Lie algebra endowed with a group homomorphism

$$S_4 \hookrightarrow \operatorname{Aut} \mathfrak{g}.$$

The action of Klein's 4-group $V = \operatorname{span}\{\tau_1, \tau_2\}$ gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading:

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Lie algebras with S_4-symmetry (cont.)

Then $A := \mathfrak{g}_0$ is an algebra with involution (the coordinate algebra of $\mathfrak{g}$):

$$xy = -\tau\big([\varphi(x), \varphi^2(y)]\big), \qquad \bar{x} = -\tau(x),$$

and there is a representation:

$$\begin{aligned} \rho : \mathfrak{t} &\longrightarrow \mathfrak{tri}(A) \\ d &\mapsto \big(\rho_0(d), \rho_1(d), \rho_2(d)\big) \\ &\qquad \Big(\rho_i(d) : x \mapsto \varphi^{-i}\big([d, \varphi^i(x)]\big)\Big) \end{aligned}$$

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Examples

- J a Jordan algebra, $\mathfrak{s} = \mathfrak{so}_3 \simeq \mathfrak{sl}_2 = ke_0 + ke_1 + ke_2$, $[e_i, e_{i+1}] = e_{i+2}$.

  Consider the TKK-construction *à la Tits*

$$\mathfrak{g} = \mathfrak{der}\, J \oplus \big(\mathfrak{s} \otimes J\big)$$

with

$$\begin{cases} [d, s \otimes x] = s \otimes d(x), \\ [s \otimes x, t \otimes y] = 2tr(st)[L_x, L_y] + [s,t] \otimes xy \quad (tr = \text{trace in } \mathfrak{sl}_2). \end{cases}$$

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Examples (cont.)

- The *Tetrahedron algebra* $\mathfrak{g}_{\boxtimes}$, which is the algebra generated by $\{X_{ij} : 0 \leq i \neq j \leq 3\}$, subject to the relations:

$$\begin{cases} X_{ij} + X_{ji} = 0, \\ [X_{ij}, X_{jk}] = 2(X_{ij} + X_{jk}) & \text{for } i \neq j \neq k \neq i \\ [X_{hi}, [X_{hi}, [X_{hi}, X_{jk}]]] = 4[X_{hi}, X_{jk}] & \text{for } \{h, i, j, k\} = \{0, 1, 2, 3\} \end{cases}$$

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Structurable algebras

Theorem

If the coordinate algebra of a Lie algebra with S_4-symmetry is unital, then it is structurable.

Conversely, any structurable algebra appears in this way.

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S_4-symmetry of J

Let C' be a composition algebra, and let $J = H_3(C', *)$ be the associated degree 3 simple Jordan algebra.

$$e_0 = \mathrm{diag}(0,0,1), \qquad e_1 = \mathrm{diag}(1,0,0), \qquad e_2 = \mathrm{diag}(0,1,0),$$

$$\iota_0(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \iota_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \quad \iota_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}.$$

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S_4-symmetry of J (cont.)

Then $S_4 \hookrightarrow \operatorname{Aut} J$ by means of

$$\tau_1 : e_i \mapsto e_i,\ \iota_0(x) \mapsto \iota_0(x),\ \iota_1(x) \mapsto -\iota_1(x),\ \iota_2(x) \mapsto -\iota_2(x),$$
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$$S_4 \hookrightarrow \operatorname{Aut} J \ \Rightarrow\ S_4 \hookrightarrow \operatorname{Aut} \mathfrak{Der}\, J \ \Rightarrow\ S_4 \hookrightarrow \operatorname{Aut} \mathcal{T}(C, J).$$

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S_4-symmetry of $\mathcal{T}(C, J)$

Theorem

The coordinate algebra of $\mathcal{T}(C, J)$ with the above action of S_4 is isomorphic to

$$\Big(C \otimes C', (a \otimes x)(b \otimes y) = ab \otimes xy, \overline{a \otimes x} = \bar{a} \otimes \bar{x}\Big).$$

Corollary

$\mathcal{T}(C, J)$ is isomorphic to $\mathfrak{g}(C, C')$.

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$\mathcal{T}(C, J)$ is isomorphic to $\mathfrak{g}(C, C')$.

Another S_4-action on $\mathcal{T}(C, J)$

There is a natural action of S_4 on the Cayley algebra

$$C = \operatorname{span}\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\},$$

where

$$e_j^2 = e_j,\ e_1 e_2 = e_2 e_1 = 0,$$
$$e_1 u_i = u_i e_2 = u_i,\ e_2 u_i = u_i e_1 = 0,\ e_1 v_i = v_i e_2 = 0,\ e_2 v_i = v_i e_1 = v_i,$$
$$u_i u_{i+1} = -u_{i+1} u_i = u_{i+2},\ v_i v_{i+1} = -v_{i+1} v_i = v_{i+2},$$
$$u_i v_j = \delta_{ij} e_1,\ v_i u_j = \delta_{ij} e_2.$$

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$$u_iu_{i+1} = -u_{i+1}u_i = u_{i+2},\ v_iv_{i+1} = -v_{i+1}v_i = v_{i+2},$$
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Another S_4-action on $\mathcal{T}(C, J)$ (cont.)

Under the S_4-action

$$C = C_{(\bar 0,\bar 0)} \oplus C_{(\bar 1,\bar 0)} \oplus C_{(\bar 0,\bar 1)} \oplus C_{(\bar 1,\bar 1)} \quad (\mathbb{Z}_2 \times \mathbb{Z}_2\text{-grading}),$$

with

$$C_{(\bar 0,\bar 0)} = ke_1 + ke_2,\ C_{(\bar 1,\bar 0)} = ku_0 + kv_0,\ C_{(\bar 0,\bar 1)} = ku_1 + kv_1,\ C_{(\bar 1,\bar 1)} = ku_2 + kv_2.$$

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Another S_4-action on $\mathcal{T}(C, J)$ (cont.)

Theorem

The coordinate algebra here is isomorphic to

$$\mathcal{A}(J) = \left\{ \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} : \alpha, \beta \in k,\ x, y \in J \right\}$$

with multiplication and involution:

$$\begin{cases} \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \begin{pmatrix} \alpha' & x' \\ y' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + tr(xy') & \alpha x' + \beta' x + y \times y' \\ \alpha' y + \beta y' + x \times x' & \beta\beta' + tr(yx') \end{pmatrix}, \\ \overline{\begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix}} = \begin{pmatrix} \beta & x \\ y & \alpha \end{pmatrix}. \end{cases}$$

$(x \times y = 2xy - tr(xy)1.)$

Another S_4-action on $\mathcal{T}(C, J)$ (cont.)

The algebras $\mathcal{A}(J)$ are related to some Freudenthal triple systems and to constructions of the exceptional simple Lie algebras in terms of some 5-gradings.

That's all. Thanks

Another S_4-action on $\mathcal{T}(C, J)$ (cont.)

The algebras $\mathcal{A}(J)$ are related to some Freudenthal triple systems and to constructions of the exceptional simple Lie algebras in terms of some 5-gradings.

That's all. Thanks