Cross products, invariants, and centralizers

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(joint work with Georgia Benkart)

Schur-Weyl duality. General linear group

$\mathbb{F}$: algebraically closed field, $\operatorname{char}\mathbb{F} = 0$.
V finite-dimensional vector space over $\mathbb{F}$.

$$\mathsf{GL}(\mathsf{V}) \quad \curvearrowright \quad \mathsf{V}^{\otimes n} \quad \curvearrowleft \quad \mathsf{S}_n$$

$$\operatorname{End}_{\mathsf{GL}(\mathsf{V})}\big(\mathsf{V}^{\otimes n}\big) = \operatorname{alg}\langle \text{action of } \mathsf{S}_n \rangle,$$

$$\operatorname{End}_{\mathsf{S}_n}\big(\mathsf{V}^{\otimes n}\big) = \operatorname{alg}\langle \text{action of } \mathsf{GL}(\mathsf{V}) \rangle.$$

Orthogonal group

Assume that now V is endowed with a nondegenerate quadratic form. Then:

$$\mathrm{End}_{\mathrm{O(V)}}\left(\mathrm{V}^{\otimes n}\right) = \mathrm{alg}\langle \text{action of } \mathrm{S}_n \text{ and of the } c_{ij}\text{'s}\rangle,$$

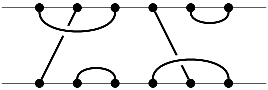
where the *contractions* are given by

$$c_{ij}(v_1 \otimes \cdots \otimes v_n) = \left(v_i \mid v_j\right) \sum_{l=1}^{r} v_1 \otimes \cdots \otimes e_l \otimes \cdots \otimes f_l \otimes \cdots \otimes v_n,$$

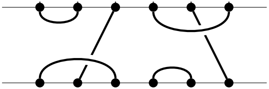
where $\left\{e_l\right\}$ and $\left\{f_l\right\}$ are dual bases.

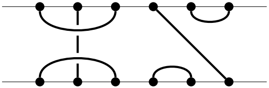
Brauer algebra

Br(x) is the algebra with basis consisting of diagrams of the form:

$\alpha =$ 

with multiplication given by *bordism*, and by multiplying by the parameter x each time we get a circle:

$\beta =$ 

$\alpha \circ \beta = x$ 

Orthogonal and symplectic groups

Orthogonal group: $\mathrm{O(V)} \quad \curvearrowright \quad \mathrm{V}^{\otimes n} \quad \curvearrowleft \quad \mathrm{Br(dim\,V)}$

$$\mathrm{End}_{\mathrm{O(V)}}\left(\mathrm{V}^{\otimes n}\right) = \mathrm{alg}\langle \text{action of } \mathrm{Br}(\dim \mathrm{V})\rangle,$$

$$\mathrm{End}_{\mathrm{Br}(\dim \mathrm{V})}\left(\mathrm{V}^{\otimes n}\right) = \mathrm{alg}\langle \text{action of } \mathrm{O(V)}\rangle.$$

Symplectic group: $\mathrm{Sp(V)} \quad \curvearrowright \quad \mathrm{V}^{\otimes n} \quad \curvearrowleft \quad \mathrm{Br}(-\dim \mathrm{V})$

$$\mathrm{End}_{\mathrm{Sp(V)}}\left(\mathrm{V}^{\otimes n}\right) = \mathrm{alg}\langle \text{action of } \mathrm{Br}(-\dim \mathrm{V})\rangle,$$

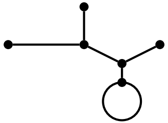
$$\mathrm{End}_{\mathrm{Br}(-\dim \mathrm{V})}\left(\mathrm{V}^{\otimes n}\right) = \mathrm{alg}\langle \text{action of } \mathrm{Sp(V)}\rangle.$$

What about G_2 and its natural representation?

$$\mathrm{G}_2 \quad \curvearrowright \quad \mathrm{V}^{\otimes n} \quad \curvearrowleft \quad ??$$

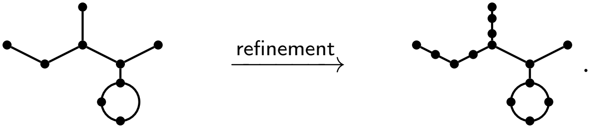
3-tangles

A **3-tangle** is an *equivalence class* of graphs with nodes of valence 1, 2 or 3, together with an orientation on the edges incident to each node of valence 3:

$\gamma =$ 

The *boundary* consists of the nodes of valence 1: $\partial\gamma$.

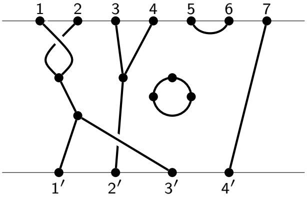
Two such graphs are said to be equivalent if they have the same boundary, and admit a common refinement. Refinements are obtained by 'splitting edges adding valence two nodes':



3-tangles $[n] \to [m]$

For $n \in \mathbb{N}$, let $[n] = \{1, \dots, n\}$ with $[0] = \emptyset$. Then, for $n, m \in \mathbb{N}$, a **3-tangle** $\gamma : [n] \to [m]$ is a 3-tangle γ with $\partial\gamma = [n] \sqcup [m]$ (disjoint union, which may thought of as $\{1, \dots, n, 1', \dots, m'\}$).

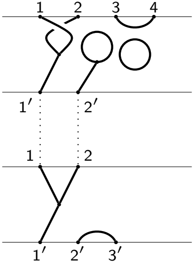
$\gamma \quad =$



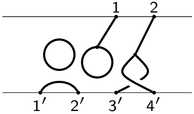
(The orientation of a valency 3 node is given by clockwise order.)

Operations on 3-tangles

$\gamma =$ / $\gamma^* =$

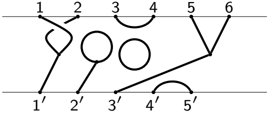


$\gamma^t =$



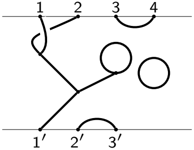
transpose

$\gamma \sqcup \gamma^* =$



disjoint union

$\gamma \circ \gamma^* =$



composition

Category $\mathcal{T}$

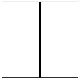
Objects: $[n]$, $n \in \mathbb{N}$ $(0 \in \mathbb{N})$.

Morphisms: linear combinations of 3-tangles $[n] \to [m]$.

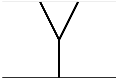
- $\sqcup$ induces a tensor product $\sqcup : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$.
- The transpose induces bijections $\mathrm{Mor}_{\mathcal{T}}([n],[m]) \to \mathrm{Mor}_{\mathcal{T}}([m],[n])$, $\gamma \mapsto \gamma^t$, such that $(\gamma^* \circ \gamma)^t = \gamma^t \circ (\gamma^*)^t$ whenever this makes sense.
- There are natural maps $\Phi_{n,m} : \mathrm{Mor}_{\mathcal{T}}([n],[m]) \longrightarrow \mathrm{Mor}_{\mathcal{T}}([n+m],[0])$ and $\Psi_{n,m} : \mathrm{Mor}_{\mathcal{T}}([n+m],[0]) \longrightarrow \mathrm{Mor}_{\mathcal{T}}([n],[m])$.

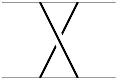
Basic morphisms

The morphisms

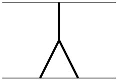
$\mathsf{I}_1 =$ 

$\beta =$ 

$\mu =$ 

$\tau =$ 

$\beta^t =$ 

$\mu^t =$ 

are called *basic*. They constitute the *alphabet* of $\mathcal{T}$.

Category $\mathcal{T}_\Gamma$

In addition to generators, some relations can be imposed in the category $\mathcal{T}$.

Let

$$\Gamma = \{\gamma_i \in \mathrm{Mor}_{\mathcal{T}}([n_i],[m_i]) : i = 1,\dots,k\}$$

be a finite set of morphisms in $\mathcal{T}$. For each $n, m \in \mathbb{N}$, the set Γ generates, through compositions and tensor products with arbitrary 3-tangles, a subspace $\mathrm{R}_\Gamma([n],[m])$ of $\mathrm{Mor}_{\mathcal{T}}([n],[m])$, and we define a new category $\mathcal{T}_\Gamma$ with the same objects and with

$$\mathrm{Mor}_{\mathcal{T}_\Gamma}([n],[m]) = \mathrm{Mor}_{\mathcal{T}}([n],[m]) / \mathrm{R}_\Gamma([n],[m]).$$

$\mathcal{T}_\Gamma$ is the 3-tangle category associated with the set of relations Γ.

The functor $\mathcal{R}_{\mathfrak{V}}$

- $\mathfrak{V} = (\mathsf{V}, \mathsf{b}, \mathsf{m})$ finite-dimensional nonassociative algebra with multiplication m, endowed with an associative, nondegenerate, symmetric bilinear form $\mathsf{b} : \mathsf{V} \times \mathsf{V} \to \mathbb{F}$.
- Let $\mathcal{V}$ be the category of finite-dimensional vector spaces over $\mathbb{F}$ with linear maps as morphisms.
- Denote by τ the switch map $\tau : \mathsf{V}^{\otimes 2} \to \mathsf{V}^{\otimes 2}$, $x \otimes y \mapsto y \otimes x$. Identify b with a linear map $\mathsf{V}^{\otimes 2} \to \mathbb{F}$ and m with a linear map $\mathsf{V}^{\otimes 2} \to \mathsf{V}$. Let 1_V be the identity map on V.

Theorem (Boos, Cadorin, Knus, Rost 1998–2005)

There exists a unique functor $\mathcal{R}_{\mathfrak{V}} : \mathcal{T} \to \mathcal{V}$ such that:

1. $\mathcal{R}_{\mathfrak{V}}([0]) = \mathbb{F}$ *and* $\mathcal{R}_{\mathfrak{V}}([n]) = \mathsf{V}^{\otimes n}$, *for any* $n \geq 1$.
2. $\mathcal{R}_{\mathfrak{V}}(\mathsf{I}_1) = 1_\mathsf{V}$ *and* $\mathcal{R}_{\mathfrak{V}}(\tau) = \tau$.
3. $\mathcal{R}_{\mathfrak{V}}(\beta) = \mathsf{b}$, $\mathcal{R}_{\mathfrak{V}}(\mu) = \mathsf{m}$, $\mathcal{R}_{\mathfrak{V}}(\gamma^t) = \mathcal{R}_{\mathfrak{V}}(\gamma)^t$, *and* $\mathcal{R}_{\mathfrak{V}}(\gamma \sqcup \delta) = \mathcal{R}_{\mathfrak{V}}(\gamma) \otimes \mathcal{R}_{\mathfrak{V}}(\delta)$, *for morphisms* γ *and* δ *in* $\mathcal{T}$.

The functor $\mathcal{R}_{\mathfrak{V}}$

Remark

$$\mathcal{R}_{\mathfrak{V}}(\beta^t \circ \beta) = \dim_{\mathbb{F}} \mathsf{V} \in \mathbb{F} \cong \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, \mathbb{F}).$$

Let

$$\Gamma_{\mathfrak{V}} = \{c_i \in \mathrm{Hom}_{\mathbb{F}}(\mathsf{V}^{\otimes n_i}, \mathsf{V}^{\otimes m_i}) : i = 1, \ldots, k\}$$

be a finite set of homomorphisms.

Definition

The algebra $\mathfrak{V}$ is said to be **of tensor type $\Gamma_{\mathfrak{V}}$** if the c_i's are identities for $\mathfrak{V}$, i.e., if

$$c_i(x_1 \otimes \cdots \otimes x_{n_i}) = 0$$

for all $i = 1, \ldots, k$, and all $x_1, \ldots, x_{n_i} \in \mathsf{V}$.

The functor $\mathcal{R}_{\mathfrak{V}}$

Corollary

- $\mathfrak{V} = (\mathsf{V}, \mathsf{b}, \mathsf{m})$ *be an algebra of tensor type* $\Gamma_{\mathfrak{V}} = \{\mathsf{c}_i : i = 1, \ldots, k\}$, *for tensors* c_i *expressible in terms of the alphabet*, 1_V, m, m^t, b, b^t, *and* τ*.*
- $\Gamma = \{\gamma_i : i = 1, \ldots, k\}$ *with* $\mathcal{R}_{\mathfrak{V}}(\gamma_i) = \mathsf{c}_i$ *for all* $i = 1, \ldots, k$*.*

Then there is a unique functor $\mathcal{R}_\Gamma : \mathcal{T}_\Gamma \to \mathcal{V}$ *such that* $\mathcal{R}_{\mathfrak{V}} = \mathcal{R}_\Gamma \circ \mathcal{P}$, *where* $\mathcal{P}$ *is the natural projection* $\mathcal{T} \to \mathcal{T}_\Gamma$*.*

Cross products

Let (V, b) be a vector space V over $\mathbb{F}$ equipped with a nondegenerate symmetric bilinear form b. A **cross product** on (V, b) is a bilinear multiplication $\mathrm{V} \times \mathrm{V} \to \mathrm{V}$, $(u, v) \mapsto u \times v$, such that:

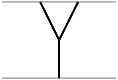
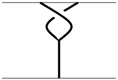
$$
\begin{aligned}
& u \times u = 0, \\
& \mathrm{b}(u \times v, u) = 0, \\
& \mathrm{b}(u \times v, u \times v) = \begin{vmatrix} \mathrm{b}(u, u) & \mathrm{b}(u, v) \\ \mathrm{b}(v, u) & \mathrm{b}(v, v) \end{vmatrix},
\end{aligned}
$$

for any $u, v \in \mathrm{V}$.

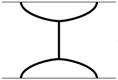
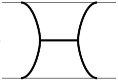
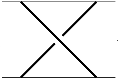
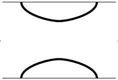
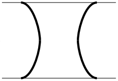
A nonzero cross product exists only if $\dim_{\mathbb{F}} \mathrm{V} = 3$ or 7.

Cross products

- Anticommutativity of the cross product corresponds, through $\mathcal{R}_{\mathfrak{Y}}$ to the identity

$\gamma_1 :$  $+$  $= 0.$

- The last condition on the definition of cross product corresponds to

$\gamma_2 :$  $+$  -2  $+$  $+$  $= 0.$

- The dimension corresponds to:

$\gamma_0 : \ \beta^t \circ \beta - (\dim_{\mathbb{F}} V)1 =$  $- (\dim_{\mathbb{F}} V)1 \quad = 0.$

Cross products

Proposition

If $\mathfrak{V} = (\mathsf{V}, \mathsf{b}, \times)$ for a vector space V endowed with a nonzero cross product $\times$ relative to the nondegenerate symmetric bilinear form b, then the functor $\mathcal{R}_{\mathfrak{V}}$ induces a functor $\mathcal{R}_{\Gamma} : \mathcal{T}_{\Gamma} \to \mathcal{V}$, with $\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}$.

7-dimensional cross product

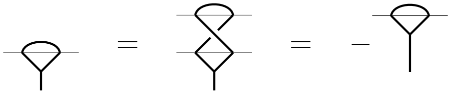
Goal

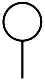
To prove that

$$\mathcal{R}_\Gamma : \mathrm{Mor}_{\mathcal{T}_\Gamma}([n],[m]) \longrightarrow \mathsf{Hom}_{\mathsf{Aut}(\mathsf{V},\times)}(\mathsf{V}^{\otimes n}, \mathsf{V}^{\otimes m})$$

is a bijection.

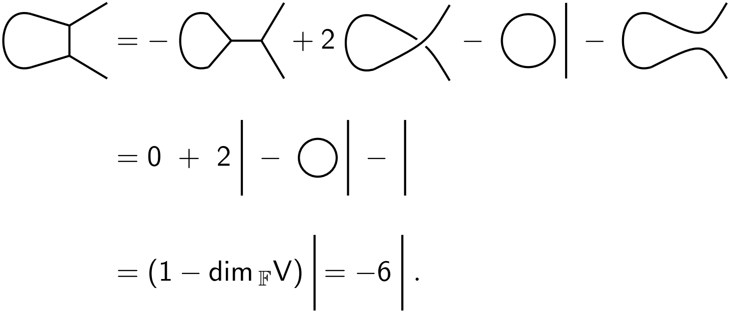
Several steps will be followed:

- 

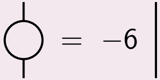
  Therefore, in $\mathcal{T}_\Gamma$ we have  $= 0$

7-dimensional cross product

- Relation γ_2 gives:

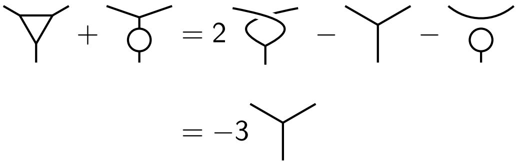


so

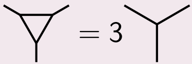


7-dimensional cross product

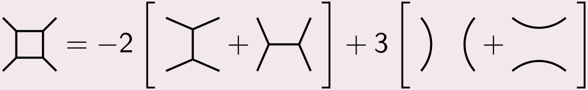
- Again with relation γ_2, we get:

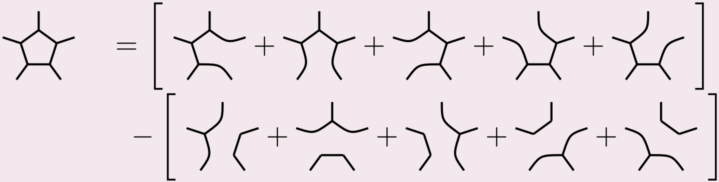


so we can get rid of triangles:



7-dimensional cross product

- 

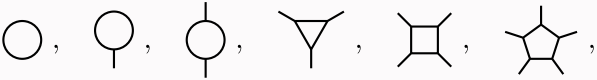
- 

7-dimensional cross product

Theorem

Let $n, m \in \mathbb{N}$*, and* $\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}$*,*

(a) *The classes modulo* Γ *of the* 3*-tangles* $[n] \to [m]$ *without crossings and without any of the subgraphs:*



form a basis of $\mathrm{Mor}_{\mathcal{T}_\Gamma}([n],[m])$*.*

(b) *The functor* $\mathcal{R}_\Gamma$ *gives a linear isomorphism*

$$\mathrm{Mor}_{\mathcal{T}_\Gamma}([n],[m]) \to \mathsf{Hom}_{\mathsf{Aut}(\mathsf{V},\times)}\big(\mathsf{V}^{\otimes n}, \mathsf{V}^{\otimes m}\big).$$

(c) *The* 3*-tangles* $[n] \to [n]$ *as in part* (a) *give a basis of the centralizer algebra:* $\mathsf{End}_{\mathsf{Aut}(\mathsf{V},\times)}\big(\mathsf{V}^{\otimes n}\big) \simeq \mathrm{Mor}_{\mathcal{T}_\Gamma}([n],[n])$*.*

3-dimensional cross product

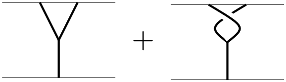
$\mathfrak{V} = (\mathsf{V}, \mathsf{b}, \times)$ a 3-dimensional vector space over $\mathbb{F}$, endowed with a nonzero cross product $u \times v$, relative to a nondegenerate symmetric bilinear form b. Then we have

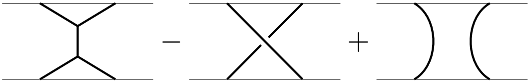
$$
\begin{aligned}
&u \times u = 0,\\
&\mathsf{b}(u \times v, u) = 0,\\
&(u \times v) \times w = \mathsf{b}(u, w)v - \mathsf{b}(v, w)u,
\end{aligned}
$$

for any $u, v, w \in \mathsf{V}$.

We must replace $\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}$ above by $\widehat{\Gamma} = \{\widehat{\gamma}_0, \widehat{\gamma}_1 = \gamma_1, \widehat{\gamma}_2\}$:

$\widehat{\gamma}_0 = \beta^t \circ \beta - 3$:  $- 3$

$\widehat{\gamma}_1 =$ 

$\widehat{\gamma}_2 =$ 

3-dimensional cross product

Theorem

Let $n, m \in \mathbb{N}$, and $\widehat{\Gamma} = \{\widehat{\gamma}_0, \widehat{\gamma}_1, \widehat{\gamma}_2\}$.

(a) *The classes modulo $\widehat{\Gamma}$ of* normalized 3-*tangles* $[n] \to [m]$ *form a basis of* $\mathrm{Mor}_{\mathcal{T}_{\widehat{\Gamma}}}([n],[m])$.

(b) $\mathcal{R}_{\widehat{\Gamma}}$ *gives a linear isomorphism*

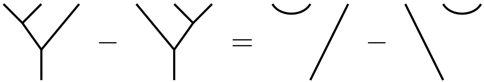
$$\mathrm{Mor}_{\mathcal{T}_{\widehat{\Gamma}}}([n],[m]) \to \mathsf{Hom}_{\mathsf{SO(V,b)}}(\mathsf{V}^{\otimes n}, \mathsf{V}^{\otimes m}).$$

(c) *The* **normalized** 3-*tangles* $[n] \to [n]$ *give a basis of the centralizer algebra* $\mathsf{End}_{\mathsf{SO(V,b)}}\big(\mathsf{V}^{\otimes n}\big) \simeq \mathrm{Mor}_{\mathcal{T}_{\widehat{\Gamma}}}([n],[n])$, *and* $\dim \mathsf{End}_{\mathsf{SO(V,b)}}\big(\mathsf{V}^{\otimes n}\big)$ *equals the number* $a(2n)$ *of Catalan partitions.*

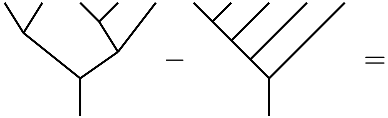
3-dimensional cross product

For the proof, as for the 7-dimensional case, we can get rid of crossings, circles, and here we get rid also of all cycles.

Relation $\widehat{\gamma}_2 = 0$ can be thought as:



which allows to prove, for instance,



integral linear combination of 'tree' 3-tangles with a lower number of trivalent nodes.

(normalized)

Kaplansky superalgebra

$$\mathrm{V}_{\bar{0}} = \mathbb{F}e, \qquad \mathrm{V}_{\bar{1}} = \mathbb{F}p \oplus \mathbb{F}q,$$

with

$$e \times e = e, \quad e \times u = u \times e = \frac{1}{2}u \quad \forall u \in \mathrm{V}_{\bar{1}},$$
$$p \times p = q \times q = 0, \quad p \times q = -q \times p = e.$$

Consider the even nondegenerate supersymmetric bilinear form $\mathrm{b} : \mathrm{V} \times \mathrm{V} \to \mathbb{F}$ such that

$$\mathrm{b}(e, e) = \frac{1}{2}, \quad \mathrm{b}(p, q) = 1.$$

Then

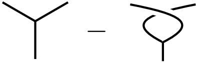
$$e \times e = e, \quad e \times u = u \times e = \frac{1}{2}u, \quad u \times v = \mathrm{b}(u, v)e,$$

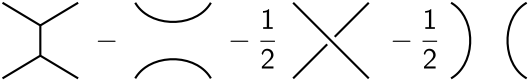
for all $u, v \in \mathrm{V}_{\bar{1}}$.

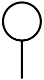
(1 | 2)-dimensional cross product

We must replace here $\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}$ by $\Gamma^s = \{\gamma_0^s, \gamma_1^s, \gamma_2^s, \gamma_3^s\}$, with

$\gamma_0^s =$  $+ 1$

$\gamma_1^s =$ 

$\gamma_2^s =$ 

$\gamma_3^s =$ 

$(1 \mid 2)$-dimensional cross product

Theorem

Let V be the 3-dimensional Kaplansky superalgebra over a field $\mathbb{F}$ of characteristic 0. Let $n, m \in \mathbb{N}$.

(a) *The classes modulo Γ^{s} of the normalized 3-tangles $[n] \to [m]$ form a basis of $\mathrm{Mor}_{\mathcal{T}_{\Gamma^{\mathsf{s}}}}([n],[m])$.*

(b) *There is a natural functor $\mathcal{R}_{\Gamma^{\mathsf{s}}}$ that gives a linear isomorphism*

$$\mathrm{Mor}_{\mathcal{T}_{\Gamma^{\mathsf{s}}}}([n],[m]) \to \mathsf{Hom}_{\mathfrak{osp}(\mathsf{V},\mathsf{b})}(\mathsf{V}^{\otimes n}, \mathsf{V}^{\otimes m}).$$

(Caution: the switch map is now $u \otimes v \mapsto (-1)^{uv} v \otimes u$.)

(c) *The normalized 3-tangles $[n] \to [n]$ give a basis of the centralizer algebra*

$$\mathsf{End}_{\mathfrak{osp}(\mathsf{V},\mathsf{b})}\left(\mathsf{V}^{\otimes n}\right) \simeq \mathrm{Mor}_{\mathcal{T}_{\Gamma^{\mathsf{s}}}}([n],[n]),$$

whose dimension is the number $a(2n)$ of Catalan partitions.