Gradings on semisimple algebras

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In 1989, Patera and Zassenhaus undertook a systematic study of gradings by abelian groups on finite-dimensional simple Lie algebras over the complex numbers, with fine gradings as the central objects. A key example of fine grading is the root space decomposition of a finite-dimensional semisimple Lie algebra relative to a Cartan subalgebra, but there are many other fine gradings that reflect the symmetries of these algebras.

A description of fine gradings on the classical simple Lie algebras (other than D_4, which is exceptional in many aspects) over $\mathbb{C}$ followed in 1998 by Havlicek, Patera, and Pelantova. The classification of fine gradings on all finite-dimensional simple Lie algebras over an algebraically closed field has been recently completed through the efforts of many authors.

Time is ripe to extend the known classifications on simple algebras to semisimple algebras.

Conventions

- The ground field $\mathbb{F}$ will be assumed to be algebraically closed.
- Only gradings by abelian groups will be considered.
- Algebras will be assumed to be finite-dimensional.
- Semisimple algebra $=$ direct sum of simple algebras.

1. Background on gradings
2. Loop algebras
3. Gradings on semisimple algebras, up to isomorphism
4. Fine group-gradings on semisimple algebras

Gradings

Definition

A grading on an algebra $\mathcal{A}$ is a set Γ of nonzero subspaces of $\mathcal{A}$ such that $\mathcal{A} = \bigoplus_{\mathcal{U}\in\Gamma} \mathcal{U}$ and for any $\mathcal{U}, \mathcal{V} \in \Gamma$, there is a $\mathcal{W} \in \Gamma$ such that $\mathcal{U}\mathcal{V} \subseteq \mathcal{W}$.

- The pair $(\mathcal{A}, \Gamma)$ is said to be a graded algebra.
- The elements of Γ are called the homogeneous components. The nonzero elements of the homogeneous components are called homogeneous elements.

Fine gradings

- Given another grading Γ' on $\mathcal{A}$, Γ is said to be a refinement of Γ' (and Γ' a coarsening of Γ) if any subspace $\mathcal{U} \in \Gamma$ is contained in a subspace in Γ'.

- The grading Γ is said to be fine if it admits no proper refinements.
  Any grading on a finite-dimensional algebra is a coarsening of a fine grading.

- Given two graded algebras $(\mathcal{A}, \Gamma)$ and $(\mathcal{A}', \Gamma')$, an equivalence $\varphi : (\mathcal{A}, \Gamma) \to (\mathcal{A}', \Gamma')$ is an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that $\varphi(\mathcal{U}) \in \Gamma'$ for each $\mathcal{U} \in \Gamma$.

Universal group

Given a graded algebra $(\mathcal{A}, \Gamma)$, consider the abelian group $U(\Gamma)$ generated by the set Γ, subject to the relations $\mathcal{U}\mathcal{V}\mathcal{W}^{-1} = e$ for each pair $\mathcal{U}, \mathcal{V}$ in Γ such that $0 \neq \mathcal{U}\mathcal{V} \subseteq \mathcal{W}$:

$$U(\Gamma) := \langle \Gamma \mid \mathcal{U}\mathcal{V}\mathcal{W}^{-1} = e \text{ if } 0 \neq \mathcal{U}\mathcal{V} \subseteq \mathcal{W} \rangle.$$

That is, $U(\Gamma)$ is the quotient of the free abelian group generated by Γ, modulo the normal subgroup generated by the elements $\mathcal{U}\mathcal{V}\mathcal{W}^{-1}$ above.

Consider also the natural map:

$$\begin{aligned} \delta_\Gamma^U : \Gamma &\longrightarrow U(\Gamma) \\ \mathcal{U} &\mapsto [\mathcal{U}], \end{aligned}$$

where $[\mathcal{U}]$ denotes the class of $\mathcal{U}$ in $U(\Gamma)$.

Definition

The pair $\left(U(\Gamma), \delta_\Gamma^U\right)$ is called the universal group of the grading Γ.

Group-gradings

Definition

Given an abelian group G, a G-grading on an algebra $\mathcal{A}$ is a triple (Γ, G, δ), where Γ is a grading on $\mathcal{A}$, and $\delta : \Gamma \to G$ is a one-to-one map, such that for any $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \Gamma$ such that $0 \neq \mathcal{U}\mathcal{V} \subseteq \mathcal{W}$, $\delta(\mathcal{U})\delta(\mathcal{V}) = \delta(\mathcal{W})$.

- The 4-tuple $(\mathcal{A}, \Gamma, G, \delta)$ is said to be a G-graded algebra. If the other components are clear from the context, we may refer simply to a G-graded algebra $\mathcal{A}$.
- Write $\mathcal{A}_g = \mathcal{U}$ if $\delta(\mathcal{U}) = g$, and $\mathcal{A}_g = 0$ otherwise. Then we get the usual expression $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$.
- The range of δ is the subset $\mathrm{Supp}_G(\Gamma) := \{g \in G : \mathcal{A}_g \neq 0\}$, which is called the support of the G-grading. Thus $\Gamma = \{\mathcal{A}_g \mid g \in \mathrm{Supp}_G(\Gamma)\}$.

Group-gradings

- Given two G-graded algebras $(\mathcal{A}, \Gamma, G, \delta)$ and $(\mathcal{A}', \Gamma', G, \delta')$, an isomorphism $\varphi : (\mathcal{A}, \Gamma, G, \delta) \to (\mathcal{A}', \Gamma', G, \delta')$ is an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that $\varphi(\mathcal{A}_g) = \mathcal{A}'_g$ for each $g \in G$.
- Given a G-graded algebra $(\mathcal{A}, \Gamma, G, \delta)$ and a group H, any group homomorphism $\beta : G \to H$ defines an H-grading $(\mathcal{A}, \Gamma', H, \delta')$ by $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}'_h$, with $\mathcal{A}'_h := \bigoplus_{\beta(g)=h} \mathcal{A}_g$ for any $h \in H$. The new grading Γ' is a coarsening of Γ. If $\pi : \Gamma \to \Gamma'$ is the corresponding surjection, then the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \delta\ } & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \beta} \\ \Gamma' & \xrightarrow{\ \delta'\ } & H \end{array}$$

is commutative. In this case, the grading (Γ', H, δ') is said to be the coarsening of (Γ, G, δ) induced by β.

Group-gradings

Definition

A grading Γ on an algebra $\mathcal{A}$ is called a group-grading if there is an abelian group G and a G-grading of the form (Γ, G, δ).

A group-grading Γ on an algebra $\mathcal{A}$ is said to be a fine group-grading if it admits no proper refinements in the class of group gradings.

Any group-grading on a finite-dimensional algebra is a coarsening of a fine group-grading.

Group-gradings

Remark

If $\operatorname{char} \mathbb{F} = 2$, then the algebra $\mathbb{F} \times \mathbb{F}$ admits a unique group-grading: the trivial one. Thus the trivial grading is a fine group-grading, but it is not a fine grading, because $\{\mathbb{F} \times 0, 0 \times \mathbb{F}\}$ is finer.

Note that $\mathbf{Aut}_{\mathbb{F}}(\mathbb{F} \times \mathbb{F}) = \mathsf{C}_2$, the constant group scheme corresponding to the cyclic group of order 2, which is not diagonalizable because $\operatorname{char} \mathbb{F} = 2$.

The group grading induced by a grading

Definition

Let Γ be a grading on the algebra $\mathcal{A}$, and let $\left(U(\Gamma), \delta_\Gamma^U\right)$ be its universal group. The coarsening Γ_{gr} defined by

$$\Gamma_{\mathrm{gr}} := \left\{ \sum_{\delta_\Gamma^U(\mathcal{U})=u} \mathcal{U} \;\middle|\; u \in \delta_\Gamma^U(\Gamma) \right\}$$

is called the group-grading induced by Γ. The grading Γ_{gr} can be realized by the $U(\Gamma)$-grading $\left(\Gamma_{\mathrm{gr}}, U(\Gamma), \delta_{\Gamma_{\mathrm{gr}}}^U\right)$, where

$$\delta_{\Gamma_{\mathrm{gr}}}^U\Big(\sum_{\delta_\Gamma^U(\mathcal{U})=u} \mathcal{U} \Big) = u$$

for any $u \in \delta_\Gamma^U(\Gamma)$.

Product gradings

Definition

Let $(\mathcal{A}^i, \Gamma^i)$ be a graded $\mathbb{F}$-algebra, $i = 1, \ldots, n$. The grading on $\mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ given by:

$$\Gamma^1 \times \cdots \times \Gamma^n := \bigcup_{i=1}^{n} \{0 \times \cdots \times \mathcal{U} \times \cdots \times 0 \mid \mathcal{U} \in \Gamma^i\}$$

is called the product grading of the Γ^i's.

The universal group-grading of the product grading is (isomorphic to) the cartesian product of the universal groups.

Product group-grading

Even if $\Gamma^1, \ldots, \Gamma^n$ are group-gradings, the product grading may fail to be so. Therefore we need a different definition of product grading for group-gradings.

Definition

Let $(\mathcal{A}^i, \Gamma^i, G^i, \delta^i)$ be a G^i-group-graded algebra, $i = 1, \ldots, n$, then the product group-grading $(\Gamma^1, G^1, \delta^1) \times \cdots \times (\Gamma^n, G^n, \delta^n)$ is the group-grading on $\mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ by the abelian group $G^1 \times \cdots \times G^n$ with:

$$\begin{aligned}
\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n\right)_{(e,\ldots,e)} &= \mathcal{A}^1_e \times \cdots \times \mathcal{A}^n_e, \\
\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n\right)_{(e,\ldots,g_i,\ldots,e)} &= 0 \times \cdots \times \mathcal{A}^i_{g_i} \times \cdots \times 0,\ \ g_i \neq e, \\
\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n\right)_{(g_1,\ldots,g_n)} &= 0,\ \text{otherwise.}
\end{aligned}$$

Product group-grading

Tongue twister

If the grading groups are the universal groups, then the product group-grading coincides with the group-grading induced by the product grading.

Free product group-gradings

Definition

Let Γ^i be a group-grading on an algebra $\mathcal{A}^i$, $i = 1, \ldots, n$. Then the group-grading $(\Gamma^1 \times \cdots \times \Gamma^n)_{\mathrm{gr}}$ on $\mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ is called the free product group-grading of the Γ^i's.

Free product group-gradings

Example

Assume $\operatorname{char}\mathbb{F} \neq 2$. Up to equivalence, there are only two fine gradings on $\mathfrak{sl}_2$:

- The Cartan grading $\Gamma^1_{\mathfrak{sl}_2}$, with universal group $\mathbb{Z}$ and homogeneous components:

$$(\mathfrak{sl}_2)_{-1} = \mathbb{F}F, \quad (\mathfrak{sl}_2)_0 = \mathbb{F}H, \quad (\mathfrak{sl}_2)_1 = \mathbb{F}E.$$

- The Pauli grading $\Gamma^2_{\mathfrak{sl}_2}$ with universal group $(\mathbb{Z}/2)^2$ and homogeneous components:

$$(\mathfrak{sl}_2)_{(\bar{1},\bar{0})} = \mathbb{F}H, \quad (\mathfrak{sl}_2)_{(\bar{0},\bar{1})} = \mathbb{F}(E+F), \quad (\mathfrak{sl}_2)_{(\bar{1},\bar{1})} = \mathbb{F}(E-F).$$

Free product group-gradings

Example (continued)

The gradings on $\mathcal{L} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ obtained as free product group-gradings of the fine gradings above are the following:

- $\left(\Gamma^1_{\mathfrak{sl}_2} \times \Gamma^1_{\mathfrak{sl}_2}\right)_{\mathrm{gr}}$ with universal group $\mathbb{Z} \times \mathbb{Z}$ and homogeneous components:

$$\begin{aligned} &\mathcal{L}_{(0,0)} = \mathbb{F}H \times \mathbb{F}H, && \\ &\mathcal{L}_{(1,0)} = \mathbb{F}E \times 0, && \mathcal{L}_{(0,1)} = 0 \times \mathbb{F}E, \\ &\mathcal{L}_{(-1,0)} = \mathbb{F}F \times 0, && \mathcal{L}_{(0,-1)} = 0 \times \mathbb{F}F. \end{aligned}$$

  (This is the Cartan grading!)

Free product group-gradings

Example (continued)

- $\left(\Gamma^1_{\mathfrak{sl}_2} \times \Gamma^2_{\mathfrak{sl}_2}\right)_{\mathrm{gr}}$ with universal group $\mathbb{Z} \times (\mathbb{Z}/2)^2$ and homogeneous components:

$$\mathcal{L}_{(0,(\bar{0},\bar{0}))} = \mathbb{F}H \times 0, \qquad \mathcal{L}_{(0,(1,\bar{0}))} = 0 \times \mathbb{F}H,$$
$$\mathcal{L}_{(1,(\bar{0},\bar{0}))} = \mathbb{F}E \times 0, \qquad \mathcal{L}_{(-1,(\bar{0},\bar{0}))} = \mathbb{F}F \times 0,$$
$$\mathcal{L}_{(0,(\bar{0},\bar{1}))} = 0 \times \mathbb{F}(E + F), \qquad \mathcal{L}_{(0,(\bar{1},\bar{1}))} = 0 \times \mathbb{F}(E - F).$$

Free product group-gradings

Example (continued)

- $\left(\Gamma^2_{\mathfrak{sl}_2} \times \Gamma^2_{\mathfrak{sl}_2}\right)_{\mathrm{gr}}$ with universal group $(\mathbb{Z}/2)^4$ and homogeneous components:

$$\begin{aligned}
\mathcal{L}_{(\bar{1},\bar{0},\bar{0},\bar{0})} &= \mathbb{F}H \times 0, & \mathcal{L}_{(\bar{0},\bar{0},\bar{1},\bar{0})} &= 0 \times \mathbb{F}H,\\
\mathcal{L}_{(\bar{0},\bar{1},\bar{0},\bar{0})} &= \mathbb{F}(E+F) \times 0, & \mathcal{L}_{(\bar{0},\bar{0},\bar{0},\bar{1})} &= 0 \times \mathbb{F}(E+F),\\
\mathcal{L}_{(\bar{1},\bar{1},\bar{0},\bar{0})} &= \mathbb{F}(E-F) \times 0 & \mathcal{L}_{(\bar{0},\bar{0},\bar{1},\bar{1})} &= 0 \times \mathbb{F}(E-F).
\end{aligned}$$

All these free product gradings are fine group-gradings, but they do not exhaust the list of fine group-gradings.

Question

What is missing?

Product G-grading

Besides the product grading, the product group-grading, and the free product group-grading, there is one more natural definition of product of gradings.

Given an abelian group G, and G-graded algebras $(\mathcal{A}^i, \Gamma^i, G, \delta^i)$, $i = 1, \dots, n$, there is a natural G-grading (Γ, G, δ) on the cartesian product $\mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ determined by

$$(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n)_g = \mathcal{A}^1_g \times \cdots \times \mathcal{A}^n_g$$

for any $g \in G$.

Definition

The G-grading above will be denoted by $\left(\mathcal{A}^1 \times \cdots \times \mathcal{A}^n, \Gamma^1 \times_G \cdots \times_G \Gamma^n, G, \delta^1 \times_G \cdots \times_G \delta^n\right)$ and will be called the product G-grading of the (Γ^i, G, δ^i)'s.

Simple algebras

Let $\mathcal{B}$ be an algebra over $\mathbb{F}$:

- $\mathcal{B}$ is simple if it has no proper ideals and $\mathcal{B}^2 \neq 0$.
  In other words, $\mathcal{B}$ is simple if it is simple as a module for its multiplication algebra $\mathrm{Mult}(\mathcal{B})$.

- The centroid of $\mathcal{B}$ is the centralizer of $\mathrm{Mult}(\mathcal{B})$ in $\mathrm{End}_{\mathbb{F}}(\mathcal{B})$:

  $$C(\mathcal{B}) := \{f \in \mathrm{End}_{\mathbb{F}}(\mathcal{B}) : f(xy) = f(x)y = xf(y)\ \forall x, y \in \mathcal{B}\}.$$

  $C(\mathcal{B})$ is commutative if $\mathcal{B}^2 = \mathcal{B}$, and it is a field (an extension field of $\mathbb{F}$) if $\mathcal{B}$ is simple.

- $\mathcal{B}$ is central simple if it is simple and central: $C(\mathcal{B}) = \mathbb{F}\mathrm{id}$.

Graded-simple algebras

Let $(\mathcal{B}, \Gamma, G, \delta)$ be a G-graded algebra: $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$.

- $\mathcal{B}$ is graded-simple if it has no proper graded ideals and $\mathcal{B}^2 \neq 0$.

  Its centroid *inherits* a G-grading:

$$C(\mathcal{B})_g := \{f \in C(\mathcal{B}) : f(\mathcal{B}_h) \subseteq \mathcal{B}_{gh} \ \forall h \in G\}.$$

- $\mathcal{B}$ is graded-central if $C(\mathcal{B})_e = \mathbb{F}\mathrm{id}$.

- $\mathcal{B}$ is graded-central-simple if it is graded-simple and graded-central.

Graded-simple algebras

Let $(\mathcal{B}, \Gamma, G, \delta)$ be a graded-simple algebra, then:

- $C(\mathcal{B})$ is a graded field (i.e., a commutative graded division algebra).
- $\mathcal{B}$ is simple (ungraded) if and only if $C(\mathcal{B})$ is a field.
- $\mathbb{K} = C(\mathcal{B})_e$ is a field, and $\mathcal{B}$ is graded-central-simple considered as an algebra over $\mathbb{K}$.
- If $\mathcal{B}$ is graded-central simple, and H is the support of the induced grading on $C(\mathcal{B})$, then $C(\mathcal{B})$ is isomorphic to the group algebra $\mathbb{F}H$, as a G-graded algebra.

Loop algebras

Definition (Allison, Berman, Faulkner, Pianzola)

Let $\pi : G \to \overline{G}$ be a surjective group homomorphism between the abelian groups G and $\overline{G}$. Given any $\overline{G}$-graded algebra $(\mathcal{A}, \overline{\Gamma}, \overline{G}, \bar{\delta})$, the associated loop algebra is the G-graded algebra $(L_\pi(\mathcal{A}), \Gamma, G, \delta)$, where

$$L_\pi(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{A}_{\pi(g)} \otimes g \quad \left(\leq \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G\right)$$

and $L_\pi(\mathcal{A})_g = \mathcal{A}_{\pi(g)} \otimes g$ for any $g \in G$.

Proposition

Let $\pi : G \to \overline{G}$ be a surjective group homomorphism of abelian groups. Let $(\mathcal{A}, \overline{\Gamma}, \overline{G}, \bar{\delta})$ be a $\overline{G}$-graded algebra, and let $(L_\pi(\mathcal{A}), \Gamma, G, \delta)$ be the associated loop algebra. Then $(\overline{G}, \bar{\delta})$ is, up to isomorphism, the universal group of $\overline{\Gamma}$ if and only if (G, δ) is, up to isomorphism, the universal group of Γ.

Theorem

1. *Let $\pi : G \to \overline{G}$ be a surjective group homomorphism with kernel H and let $(\mathcal{A}, \overline{\Gamma}, \overline{G}, \bar{\delta})$ be a central simple $\overline{G}$-graded algebra. Then the associated loop algebra $(L_\pi(\mathcal{A}), \Gamma, G, \delta)$ is graded-central-simple and the map*

$$\begin{aligned} \mathbb{F}H &\longrightarrow C\big(L_\pi(\mathcal{A})\big) \\ h &\mapsto \Big(x \otimes g \mapsto x \otimes hg\Big) \end{aligned}$$

for any $g \in G$ and $x \in \mathcal{A}_{\pi(g)}$, is an isomorphism of G-graded algebras.

Theorem (continued)

2. *Let* $(\mathcal{B}, \tilde{\Gamma}, G, \tilde{\delta})$ *be a graded-central-simple* G*-graded algebra, and* $H = \mathrm{Supp}_G(\Gamma_{C(\mathcal{B})})$*. Let* $\pi : G \to \overline{G}$ *be a surjective group homomorphism with kernel* H*. Then there exists a central simple* $\overline{G}$*-graded algebra* $(\mathcal{A}, \overline{\Gamma}, \overline{G}, \bar{\delta})$ *such that* $(\mathcal{B}, \tilde{\Gamma}, G, \tilde{\delta})$ *is isomorphic, as a* G*-graded algebra, to the associated loop algebra* $(L_\pi(\mathcal{A}), \Gamma, G, \delta)$*.*
   Moreover, the algebra $\mathcal{A}$ *is a quotient of* $\mathcal{B}$*.*

Theorem (continued)

3. *Let H^1 and H^2 be subgroups of G, consider the quotient groups $\overline{G}^i = G/H^i$ and the natural projections $\pi^i : G \to \overline{G}^i$, $i = 1, 2$. Let $(\mathcal{A}^i, \overline{\Gamma}^i, \overline{G}^i, \bar{\delta}^i)$ be a central simple $\overline{G}^i$-graded algebra for $i = 1, 2$. Then the associated loop algebras $(L_{\pi^1}(\mathcal{A}^1), \Gamma^1, G, \delta^1)$ and $(L_{\pi^2}(\mathcal{A}^2), \Gamma^2, G, \delta^2)$ are isomorphic, as G-graded algebras, if and only if $H^1 = H^2$ and the $\overline{G} = G/H^1$-graded algebras $(\mathcal{A}^1, \overline{\Gamma}^1, \overline{G}^1, \bar{\delta}^1)$ and $(\mathcal{A}^2, \overline{\Gamma}^2, \overline{G}^2, \bar{\delta}^2)$ are isomorphic.*

Theorem

Let $\pi : G \to \overline{G}$ be a surjective group homomorphism of abelian groups with finite kernel $H = \ker \pi$. Let $(\mathcal{A}, \overline{\Gamma}, \overline{G}, \bar{\delta})$ be a central simple $\overline{G}$-graded algebra and let $(L_\pi(\mathcal{A}), \Gamma, G, \delta)$ be the associated loop algebra. Then $L_\pi(\mathcal{A})$ is semisimple if and only if the characteristic of $\mathbb{F}$ does not divide the order of H.
If this is the case, then $L_\pi(\mathcal{A})$ is isomorphic to the cartesian product of $|H|$ copies of $\mathcal{A}$.

Gradings on semisimple algebras, up to isomorphism

Theorem

1. *Let* $(\mathcal{B}, \Gamma, G, \delta)$ *be a semisimple* G*-graded algebra, then* $(\mathcal{B}, \Gamma, G, \delta)$ *is isomorphic, as a* G*-graded algebra, to a product* G*-grading*

$$\left(\mathcal{B}^1 \times \cdots \times \mathcal{B}^n, \Gamma^1 \times_G \cdots \times_G \Gamma^n, G, \delta^1 \times_G \cdots \times_G \delta^n\right)$$

for some graded-simple and semisimple G*-graded algebras* $(\mathcal{B}^i, \Gamma^i, G, \delta^i)$*,* $i = 1, \ldots, n$*.*
The factors $(\mathcal{B}^i, \Gamma^i, G, \delta^i)$ *are uniquely determined up to reordering and* G*-graded isomorphisms.*

Gradings on semisimple algebras, up to isomorphism

Theorem (continued)

2. *Any finite-dimensional graded-simple G-graded algebra* $(\mathcal{B}, \tilde{\Gamma}, G, \tilde{\delta})$ *is isomorphic, as a G-graded algebra, to the loop algebra* $(L_\pi(\mathcal{A}), \Gamma, G, \delta)$ *associated to a surjective group homomorphism* $\pi : G \to \overline{G}$ *with finite kernel H, and a central simple* $\overline{G}$*-graded algebra* $(\mathcal{A}, \overline{\Gamma}, \overline{G}, \bar{\delta})$*. The algebra* $\mathcal{A}$ *is a quotient of* $\mathcal{B}$*.*
   Moreover, in this situation $\mathcal{B}$ *is semisimple if and only if* $\operatorname{char} \mathbb{F}$ *does not divide the order of H.*

Fine gradings on semisimple algebras

Fine (general) gradings behave very well with respect to cartesian products:

Proposition

Let Γ be a fine grading on an algebra which is a direct sum of graded ideals $\mathcal{B} = \mathcal{B}^1 \oplus \cdots \oplus \mathcal{B}^n$. Let $\Gamma^i = \Gamma|_{\mathcal{B}^i}$ be the induced grading on $\mathcal{B}^i$ for each $i = 1, \ldots, n$. Then each Γ^i is a fine grading, and Γ is equivalent to the product grading $\Gamma|_{\mathcal{B}^1} \times \cdots \times \Gamma|_{\mathcal{B}^n}$ on $\mathcal{B}^1 \times \cdots \times \mathcal{B}^n$ (naturally isomorphic to $\mathcal{B}$). Conversely, let Γ^i be a fine grading on $\mathcal{B}^i$, for $i = 1, \ldots, n$, then the product grading $\Gamma^1 \times \cdots \times \Gamma^n$ is a fine grading on $\mathcal{B}^1 \times \cdots \times \mathcal{B}^n$.

For fine group-gradings, the situation is a bit more involved.

Fine group-gradings on semisimple algebras

Theorem

1. *Any fine group-grading on a finite-dimensional semisimple algebra is equivalent to a free product group-grading* $(\Gamma^1 \times \cdots \times \Gamma^n)_{\mathrm{gr}}$*, with the* Γ^i*'s being fine group-gradings on a semisimple graded-simple algebra* $\mathcal{B}^i$*, satisfying one of the following extra conditions:*
   - char $\mathbb{F} = 2$ *and for any index* i *such that* Γ^i *is trivial, there is at most one other index* j *such that* $(\mathcal{B}^i, \Gamma^i)$ *is equivalent to* $(\mathcal{B}^j, \Gamma^j)$*.*
   - char $\mathbb{F} \neq 2$ *and for any index* i *such that* Γ^i *is trivial, there is no other index* j *such that* $(\mathcal{B}^i, \Gamma^i)$ *is equivalent to* $(\mathcal{B}^j, \Gamma^j)$*.*

   And conversely, any such free product group-grading is a fine group-grading.

   Moreover, the factors $(\mathcal{B}^i, \Gamma^i)$ *are uniquely determined, up to reordering and equivalence.*

Fine group-gradings on semisimple algebras

Theorem (continued)

2. *Any finite-dimensional graded-simple algebra* $(\mathcal{B}, \Gamma')$ *such that* Γ' *is a fine group-grading is equivalent to a loop algebra* $(L_\pi(\mathcal{A}), \Gamma, U, \delta)$ *associated to a surjective group homomorphism* $\pi : U \to \overline{U}$ *with finite kernel, and a simple finite-dimensional graded algebra* $(\mathcal{A}, \overline{\Gamma}, \overline{U}, \bar{\delta})$ *with* $\overline{\Gamma}$ *a fine group-grading with universal group* $(\overline{U}, \bar{\delta})$.
   And conversely.
   Moreover, in this situation $\mathcal{B}$ *is semisimple if and only if* $\operatorname{char} \mathbb{F}$ *does not divide the order of* $\ker \pi$.

Fine group-gradings on semisimple algebras

Theorem (continued)

3. *For $i = 1, 2$, let $(\mathcal{A}^i, \overline{\Gamma}^i, \overline{U}^i, \bar{\delta}^i)$ consist of a simple algebra $\mathcal{A}^i$ endowed with a fine group-grading $\overline{\Gamma^i}$ with universal group $(\overline{U}^i, \bar{\delta}^i)$, and let $\pi^i : U^i \to \overline{U}^i$ be a surjective group homomorphism. Let $(L_{\pi^i}(\mathcal{A}^i), \Gamma^i, U^i, \delta^i)$ be the associated loop algebra. Then the group-graded algebras $(L_{\pi^1}(\mathcal{A}^1), \Gamma^1)$ and $(L_{\pi^2}(\mathcal{A}^2), \Gamma^2)$ are equivalent if and only if there is an equivalence $\varphi : (\mathcal{A}^1, \overline{\Gamma}^1) \to (\mathcal{A}^2, \overline{\Gamma}^2)$ such that the associated group isomorphism $\alpha_\varphi^U : \overline{U}^1 \to \overline{U}^2$ extends to a group isomorphism $\tilde{\alpha}_\varphi^U : U^1 \to U^2$.*

Fine group-gradings on semisimple algebras

Example

Assume $\operatorname{char}\mathbb{F} \neq 2$. Up to equivalence, the fine gradings on $\mathcal{L} = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ are:

- The free product group-gradings $(\Gamma^1_{\mathfrak{sl}_2} \times \Gamma^1_{\mathfrak{sl}_2})_{\mathrm{gr}}$, $(\Gamma^1_{\mathfrak{sl}_2} \times \Gamma^2_{\mathfrak{sl}_2})_{\mathrm{gr}}$, and $(\Gamma^2_{\mathfrak{sl}_2} \times \Gamma^2_{\mathfrak{sl}_2})_{\mathrm{gr}}$, with respective universal groups $\mathbb{Z}^2$, $\mathbb{Z} \times \left(\mathbb{Z}/2\right)^2$, and $\left(\mathbb{Z}/2\right)^4$, that were considered before.
- Three more fine group-gradings obtained through the loop algebra construction:
  - $\Gamma^1_{\mathcal{L}}(\mathbb{Z} \times \mathbb{Z}/2, (0, \bar{1}), 1)$ obtained from the Cartan grading $\Gamma^1_{\mathfrak{sl}_2}$ on $\mathfrak{sl}_2$.
  - $\Gamma^2_{\mathcal{L}}\left(\left(\mathbb{Z}/2\right)^3, (\bar{0}, \bar{0}, \bar{1}), \left(\mathbb{Z}/2\right)^2\right)$ and $\Gamma^2_{\mathcal{L}}(\mathbb{Z}/4 \times \mathbb{Z}/2, (\widehat{2}, \bar{0}), \mathbb{Z}/2 \times \mathbb{Z}/2)$, obtained from the Pauli grading $\Gamma^2_{\mathfrak{sl}_2}$ on $\mathfrak{sl}_2$.

In conclusion ...

Group-gradings on semisimple algebras appear as a combination of:

- different kinds of product gradings, and
- gradings obtained by means of the loop algebra construction.

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Thank you!