Gradings on simple Lie algebras

Alberto Elduque

Universidad de Zaragoza

(based on joint work with Mikhail Kochetov)

Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading (r being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than D_4, by arbitrary abelian groups were considered by Havlícek, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including D_4) over algebraically closed fields of characteristic zero has been obtained quite recently.

For any abelian group G, the classification of all G-gradings, up to isomorphism, on the classical simple Lie algebras other than D_4 over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

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Gradings on the octonions, on the Albert algebra, and on some other algebraic structures, are instrumental in obtaining a classification of the gradings on the exceptional simple Lie algebras.

Outline

Gradings:

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Exceptional algebras:

4. Composition algebras
5. Gradings on octonions and G_2
6. Gradings on the Albert Algebra and F_4

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Triality and gradings on D_4:

Gradings

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Examples

Cartan grading:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$$

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This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank} \mathfrak{g}$.

Examples

Pauli matrices: $\mathcal{A} = \mathsf{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

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This grading induces a grading on $\mathfrak{sl}_n(\mathbb{F})$.

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The grading Γ is then a grading too by $U(\Gamma)$.

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- The quotient $W(\Gamma) = \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$ is the Weyl group of Γ.

$W(\Gamma)$ acts by automorphisms on $U(\Gamma)$

Each $\varphi \in \mathsf{Aut}(\Gamma)$ determines a self-bijection α of $\mathrm{Supp}\,\Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\mathsf{Aut}(\Gamma) \to \mathsf{Aut}(U(\Gamma))$$

with kernel $\mathrm{Stab}(\Gamma)$.

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Remark

$\mathrm{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$.

Fine gradings

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Remark

Any grading is a coarsening of a fine grading.

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$\mathbb{F} = \overline{\mathbb{F}}$, $\operatorname{char} \mathbb{F} = 0$, $\dim \mathcal{A} < \infty$.

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The homogeneous components are the eigenspaces for the action of $\hat{G}$!!

MAD subgroups

Theorem

(Fine) gradings (up to equivalence)	$\longleftrightarrow$	*(maximal) abelian diagonalizable subgroups of* $\mathrm{Aut}(\mathcal{A})$ *(up to conjugation)*

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The complete classification of the fine gradings up to equivalence on the classical Lie algebras (including D_4) was obtained in 2010.

Exceptional algebras

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- E_7, E_8: Recent work by Jun Yu classifying conjugacy classes of certain subgroups of the compact Lie groups classifies, in particular, the fine gradings on E_7 and E_8 over $\mathbb{C}$.
  This is enough to classify these gradings over arbitrary algebraically closed fields of characteristic 0 (E. 2014).

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Theorem

G-grading $\longleftrightarrow$ comodule algebra over the group algebra $\mathbb{F}G$.

Gradings and comodule algebras

A comodule algebra map

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induces a *generic automorphism* of $\mathbb{F}G$-algebras

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All the information on the grading Γ attached to η is contained in this single automorphism!

Affine group schemes and gradings

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where

$$\begin{aligned} G^D : \mathrm{Alg}_{\mathbb{F}} &\longrightarrow \mathrm{Grp} \\ R &\mapsto G^D(R) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R) \simeq \mathrm{Hom}_{\mathrm{Grp}}(G, R^\times), \end{aligned}$$

$$\begin{aligned} \mathbf{Aut}(\mathcal{A}) : \mathrm{Alg}_{\mathbb{F}} &\longrightarrow \mathrm{Grp} \\ R &\mapsto \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes_{\mathbb{F}} R). \end{aligned}$$

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$$\rho_R(f)\big(x_g \otimes r\big) = x_g \otimes f(g)r$$

for $f \in G^D(R) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R)$, $x_g \in \mathcal{A}_g$ and $r \in R$.

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Conversely,

$$\rho : G^D \to \mathbf{Aut}(\mathcal{A}) \quad \Longrightarrow \quad \eta : \mathcal{A} \hookrightarrow \mathcal{A} \otimes \mathbb{F}G \xrightarrow{\rho_{\mathbb{F}G}(\mathrm{id})} \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G.$$

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Message:

It is not enough to deal with $\hat{G}$ and $\operatorname{Aut}\mathcal{A}$, but also with their extensions to unital commutative and associative $\mathbb{F}$-algebras.

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Given a morphism $\mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$, any grading on $\mathcal{A}$ induces a grading on $\mathcal{B}$.

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Example

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Example

$$\mathrm{Ad} : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{A})\big).$$

If $\mathbf{Aut}(\mathcal{A}) \cong \mathbf{Aut}(\mathcal{B})$, the problems of classifying fine gradings on $\mathcal{A}$ and on $\mathcal{B}$ up to equivalence (or the problem of classifying gradings up to isomorphism) are equivalent.

Classical Lie algebras

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- A_n:

$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}\big(M_r(\mathbb{F})^{(+)}\big),$$

("Affine group scheme of automorphisms and antiautomorphisms of the matrix algebra")

Classical Lie algebras

Gradings on matrix algebras (with involution) have been dealt with by Bahturin et al.

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Question

D_4 in the modular case?

Composition algebras

Definition

A composition algebra over a field $\mathbb{F}$ is a triple $(C, \cdot, n)$ where

- C is a vector space over $\mathbb{F}$,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to \mathbb{F}$ is a multiplicative nondegenerate quadratic form:
  - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate,
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The unital composition algebras will be called Hurwitz algebras.

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

$$x^{\cdot 2} - n(x,1)x + n(x)1 = 0$$

for any x.
They are endowed with an antiautomorphism, the standard conjugation:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

Cayley-Dickson doubling process

Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let λ be a nonzero scalar in the ground field $\mathbb{F}$. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$(a + bu) \cdot (c + du) = (a \cdot c + \lambda \bar{d} \cdot b) + (d \cdot a + b \cdot \bar{c})u,$$
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Notation: $CD(A, \mu, \lambda) := CD\big(CD(A, \mu), \lambda\big).$

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following:

(i) *The ground field $\mathbb{F}$ if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

Symmetric composition algebras

Definition

A composition algebra $(S, *, n)$ is said to be symmetric if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in S$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in S$.

Examples

- Para-Hurwitz algebras: Given a Hurwitz algebra $(C, \cdot, n)$, its para-Hurwitz counterpart is the composition algebra $(C, \bullet, n)$, where
$$x \bullet y = \bar{x} \cdot \bar{y}.$$
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- Okubo algebras: Assume $\operatorname{char} \mathbb{F} \neq 3$ and $\exists\, \omega \neq 1 = \omega^3$ in $\mathbb{F}$. Consider the algebra A_0 of zero trace elements in a central simple degree 3 associative algebra with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

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and norm $n(x) = -\frac{1}{2} \operatorname{tr}(x^2)$.
(There is a more general definition valid over arbitrary fields.)

Classification

Theorem (E.-Myung 93, E. 97)

Any symmetric composition algebra is either:

- *a para-Hurwitz algebra,*
- *a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),*
- *an Okubo algebra.*

The octonions

From now on, the ground field will be assumed to be algebraically closed of characteristic $\neq 2$.

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Cayley-Dickson process:

$$\begin{aligned} \mathbb{K} &= \mathbb{F} \oplus \mathbb{F}\,\mathbf{i}, & \mathbf{i}^2 &= 1, \\ \mathbb{H} &= \mathbb{K} \oplus \mathbb{K}\,\mathbf{j}, & \mathbf{j}^2 &= 1, \\ \mathbb{O} &= \mathbb{H} \oplus \mathbb{H}\,\mathbf{l}, & \mathbf{l}^2 &= 1, \end{aligned}$$

$\mathbb{O}$ is $\mathbb{Z}_2^3$-graded with

$$\deg(\mathbf{i}) = (\bar{1}, \bar{0}, \bar{0}), \quad \deg(\mathbf{j}) = (\bar{0}, \bar{1}, \bar{0}), \quad \deg(\mathbf{l}) = (\bar{0}, \bar{0}, \bar{1}).$$

Cartan grading on the Octonions

$\mathbb{O}$ contains canonical bases:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

with

$n(e_1, e_2) = n(u_i, v_i) = 1,\quad$ otherwise 0.

$$e_1^2 = e_1, \quad e_2^2 = e_2,$$
$$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$$
$$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$$
$$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2},\ v_i v_{i+1} = -v_{i+1} v_i = u_{i+2},\ \text{(indices modulo 3)}$$

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otherwise 0.

The Cartan grading is the $\mathbb{Z}^2$-grading determined by:

$$\deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1).$$

Fine gradings on the Octonions

Theorem (E. 1998)

Up to equivalence, the fine gradings on $\mathbb{O}$ are

- *the Cartan grading, and*
- *the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.*

Fine gradings on the Octonions

Sketch of proof:

Fine gradings on the Octonions

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- The Cayley-Hamilton equation: $x^2 - n(x,1)x + n(x)1 = 0$, implies that the norm has a good behavior relative to the grading:

$$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$

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- If there is a $g \in \mathrm{Supp}\,\Gamma$ with either order > 2 or $\dim \mathbb{O}_g \geq 2$, there are elements $x \in \mathbb{O}_g$, $y \in \mathbb{O}_{g^{-1}}$ with $n(x) = 0 = n(y)$, $n(x,y) = 1$. Then $e_1 = x\bar{y}$ and $e_2 = y\bar{x}$ are orthogonal primitive idempotents in $\mathbb{O}_e$, and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.

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- Otherwise $\dim \mathbb{O}_g = 1$ and $g^2 = e$ for any $g \in \mathrm{Supp}\,\Gamma$. We get the $\mathbb{Z}_2^3$-grading.

$\mathbb{Z}_2^3$-grading: Octonions as a twisted group algebra

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Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

$$\mathbb{O} = \mathbb{F}_\sigma[\mathbb{Z}_2^3],$$

where

$$e^\alpha e^\beta = \sigma(\alpha, \beta) e^{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{Z}_2^3$*, with*

$$\sigma(\alpha, \beta) = (-1)^{\psi(\alpha,\beta)},$$

$$\psi(\alpha, \beta) = \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3 + \sum_{i \le j} \alpha_i\beta_j.$$

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This allows to consider the algebra of octonions as an "associative algebra in a suitable category".

Cartan grading: Weyl group

Let S be the vector subspace spanned by $(1, 1, 1)$ in $\mathbb{R}^3$ and consider the two-dimensional real vector space $E = \mathbb{R}^3/S$. Take the elements

$$\epsilon_1 = (1, 0, 0) + S, \ \epsilon_2 = (0, 1, 0) + S, \ \epsilon_3 = (0, 0, 1) + S.$$

The subgroup $G = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3$ is isomorphic to $\mathbb{Z}^2$, and we may think of the Cartan grading Γ on the octonions $\mathbb{O}$ as the grading in which

$$\begin{aligned} &\deg(e_1) = 0 = \deg(e_2), \\ &\deg(u_i) = \epsilon_i = -\deg(v_i), \ i = 1, 2, 3. \end{aligned}$$

Cartan grading: Weyl group

Then $\mathrm{Supp}\ \Gamma = \{0\} \cup \{\pm\epsilon_i \mid i = 1, 2, 3\}$ and G is the universal group.
The set

$$\Phi := \Big(\mathrm{Supp}\ \Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \mathrm{Supp}\ \Gamma, \alpha \neq \pm\beta\}\Big) \setminus \{0\}$$

is the root system of type G_2.

Cartan grading: Weyl group

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\mathrm{Aut}(G)$, and this with a subgroup of $GL(E)$, we have:

$$\begin{aligned} W(\Gamma) &\subset \{\mu \in \mathrm{Aut}(G) \mid \mu(\mathrm{Supp}\ \Gamma) = \mathrm{Supp}\ \Gamma\} \\ &\subset \{\mu \in GL(E) \mid \mu(\Phi) = \Phi\} =: \mathrm{Aut}\,\Phi. \end{aligned}$$

The latter group is the automorphism group of the root system Φ, which coincides with its Weyl group.

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The latter group is the automorphism group of the root system Φ, which coincides with its Weyl group.

Theorem

Let Γ be the Cartan grading on the octonions. Identify $\mathrm{Supp}\ \Gamma \setminus \{0\}$ with the short roots in the root system Φ of type G_2. Then $W(\Gamma) = \mathrm{Aut}\,\Phi$.

$\mathbb{Z}_2^3$-grading: Weyl group

Theorem

Let Γ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \mathrm{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

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Remark

As any $\varphi \in \mathrm{Stab}(\Gamma)$ multiplies each of the elements **i**, **j**, **l** by either 1 or -1, we see that $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^3$. Therefore, the group $\mathrm{Aut}(\Gamma)$ is a (non-split) extension of $\mathbb{Z}_2^3$ by $W(\Gamma) \cong GL_3(2)$.

Gradings on para-Hurwitz algebras

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Theorem

Gradings on para-Hurwitz algebras of dimension 4 *or* 8

$$\updownarrow$$

Gradings on their Hurwitz counterparts.

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Gradings on their Hurwitz counterparts.

Therefore, any para-Cayley algebra is endowed with a $\mathbb{Z}_2^3$-grading.

Gradings on Okubo algebras

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Assuming $\mathbb{F}$ is a field of characteristic $\neq 3$ containing a primitive third root ω of 1, then the matrix algebra $\mathrm{Mat}_3(\mathbb{F})$ is generated by the order 3 matrices:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and the assignment

$$\deg(x) = (\bar{1}, \bar{0}), \qquad \deg(y) = (\bar{0}, \bar{1}),$$

gives a $\mathbb{Z}_3^2$-grading of $\mathrm{Mat}_3(\mathbb{F})$, which is inherited by the Okubo algebra $\bigl(\mathfrak{sl}_3(\mathbb{F}), *, n\bigr)$.

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Over algebraically closed fields, any grading on an Okubo algebra is a coarsening of either the natural $\mathbb{Z}^2$-grading (Cartan grading) or this $\mathbb{Z}_3^2$-grading.

$\mathbb{Z}_3^2$-grading

Consider the order three automorphism τ of $\mathbb{O}$:

$$\tau(e_i) = e_i, \; i = 1, 2, \quad \tau(u_j) = u_{j+1}, \; \tau(v_j) = v_{j+1}, \; j = 1, 2, 3,$$

and define a new multiplication on $\mathbb{O}$:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

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It turns out that this is too the (split) Okubo algebra, defined in a characteristic free way, and the $\mathbb{Z}_3^2$-grading is now given by setting

$$\deg e_1 = (\bar{1}, \bar{0}) \quad \text{and} \quad \deg u_1 = (\bar{0}, \bar{1}).$$

$\mathbb{Z}_3^2$-grading

	e_1	e_2	u_1	v_1	u_2	v_2	u_3	v_3
e_1	e_2	0	0	$-v_3$	0	$-v_1$	0	$-v_2$
e_2	0	e_1	$-u_3$	0	$-u_1$	0	$-u_2$	0
u_1	$-u_2$	0	v_1	0	$-v_3$	0	0	$-e_1$
v_1	0	$-v_2$	0	u_1	0	$-u_3$	$-e_2$	0
u_2	$-u_3$	0	0	$-e_1$	v_2	0	$-v_1$	0
v_2	0	$-v_3$	$-e_2$	0	0	u_2	0	$-u_1$
u_3	$-u_1$	0	$-v_2$	0	0	$-e_1$	v_3	0
v_3	0	$-v_1$	0	$-u_2$	$-e_2$	0	0	u_3

Multiplication table of the (split) Okubo algebra

Gradings on G_2

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If the characteristic of the ground field $\mathbb{F}$ is $\neq 2, 3$, then

$$\mathrm{Ad} : \textbf{Aut}\, \mathbb{O} \to \textbf{Aut}\, \mathfrak{g}_2$$

is an isomorphism.

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Theorem

Up to equivalence, the fine gradings on $\mathfrak{g}_2$ are

- *the Cartan grading, and*
- *a $\mathbb{Z}_2^3$-grading with $(\mathfrak{g}_2)_0 = 0$ and where $(\mathfrak{g}_2)_g$ is a Cartan subalgebra of $\mathfrak{g}_2$ for any $0 \neq g \in \mathbb{Z}_2^3$.*

Albert algebra

$$\mathbb{A} = H_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}, \ a_1, a_2, a_3 \in \mathbb{O} \right\}$$

$$= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}),$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \ \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \ \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Albert algebra

The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$.

Then E_i are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \quad \iota_i(a) \circ \iota_i(b) = 2n(a, b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3, where $a \bullet b = \bar{a}\bar{b}$ is the para-Hurwitz multiplication.

Cartan grading

Consider the following elements in $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$a_1 = (1,0,0,0), \qquad a_2 = (0,1,0,0), \qquad a_3 = (-1,-1,0,0),$$
$$g_1 = (0,0,1,0), \qquad g_2 = (0,0,0,1), \qquad g_3 = (0,0,-1,-1).$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

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Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

Take a canonical basis of the octonions. The assignment

$$\deg e_1 = \deg e_2 = 0, \quad \deg u_i = g_i = -\deg v_i$$

gives the Cartan grading on $\mathbb{O}$.

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Now, the Cartan grading on $\mathbb{A}$ is given by:

$$\begin{aligned}
&\deg E_i = 0,\ \deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2),\\
&\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i),\\
&\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}),\\
&\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}).
\end{aligned}$$

Cartan grading: Weyl group

The universal group of the Cartan grading is $\mathbb{Z}^4$, which is contained in $E = \mathbb{R}^4$. Consider the following elements of $\mathbb{Z}^4$:

$$\begin{aligned}
\epsilon_0 &= \deg \iota_1(e_1) = a_1 = (1,0,0,0),\\
\epsilon_1 &= \deg \iota_1(u_1) = g_1 = (0,0,1,0),\\
\epsilon_2 &= \deg \iota_1(u_2) = a_3 + g_2 = (-1,-1,0,1),\\
\epsilon_3 &= \deg \iota_1(u_3) = -a_2 + g_3 = (0,-1,-1,-1).
\end{aligned}$$

Note that the ϵ_i's, $0 \leq i \leq 3$, are linearly independent, but do not form a basis of $\mathbb{Z}^4$. For instance,

$$\begin{aligned}
\deg \iota_2(e_1) = a_2 = \frac{1}{2}(-\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3),\\
\deg \iota_3(e_1) = a_3 = \frac{1}{2}(-\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3).
\end{aligned}$$

Cartan grading: Weyl group

The supports of the Cartan grading Γ on each of the subspaces $\iota_i(\mathbb{O})$ are:

$$\begin{aligned}
\mathrm{Supp}\ \iota_1(\mathbb{O}) &= \{\pm\epsilon_i \mid 0 \leq i \leq 3\},\\
\mathrm{Supp}\ \iota_2(\mathbb{O}) &= \mathrm{Supp}\ \iota_1(\mathbb{O})(\iota_3(e_1) + \iota_3(e_2))\\
&= \left\{\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{even number of + signs}\right\},\\
\mathrm{Supp}\ \iota_3(\mathbb{O}) &= \mathrm{Supp}\ \iota_1(\mathbb{O})(\iota_2(e_1) + \iota_2(e_2))\\
&= \left\{\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{odd number of + signs}\right\}.
\end{aligned}$$

Cartan grading: Weyl group

$$\begin{aligned}\Phi &:= \Big(\mathrm{Supp}\,\Gamma \cup \{\alpha+\beta \mid \alpha,\beta \in \mathrm{Supp}\,\iota_1(\mathbb{O}),\ \alpha \neq \pm\beta\}\Big) \setminus \{0\}\\ &= \mathrm{Supp}\,\iota_1(\mathbb{O}) \cup \mathrm{Supp}\,\iota_2(\mathbb{O}) \cup \mathrm{Supp}\,\iota_3(\mathbb{O})\\ &\qquad \cup \{\pm\epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3\},\end{aligned}$$

is the root system of type F_4. (Note that the ϵ_i's, $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ.)

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is the root system of type F_4. (Note that the ϵ_i's, $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ.)

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\mathrm{Aut}(\mathbb{Z}^4)$, and this with a subgroup of $GL(E)$, we have:

Theorem

Let Γ be the Cartan grading on the Albert algebra. Identify $\mathrm{Supp}\,\Gamma \setminus \{0\}$ with the short roots in the root system Φ of type F_4. Then $W(\Gamma) = \mathrm{Aut}\,\Phi$.

$\mathbb{Z}_2^5$-grading

$\mathbb{A}$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathbb{A}_{(\bar{0},\bar{0})} = \mathbb{F}E_1 + \mathbb{F}E_2 + \mathbb{F}E_3,$$

$$\mathbb{A}_{(\bar{1},\bar{0})} = \iota_1(\mathbb{O}), \qquad \mathbb{A}_{(\bar{0},\bar{1})} = \iota_2(\mathbb{O}), \qquad \mathbb{A}_{(\bar{1},\bar{1})} = \iota_3(\mathbb{O}).$$

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This $\mathbb{Z}_2^2$-grading may be combined with the fine $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ to obtain a fine $\mathbb{Z}_2^5$-grading:

$$\deg E_i = (\bar{0},\bar{0},\bar{0},\bar{0},\bar{0}),\ \ i = 1,2,3,$$
$$\deg \iota_1(x) = (\bar{1},\bar{0},\deg x),$$
$$\deg \iota_2(x) = (\bar{0},\bar{1},\deg x),$$
$$\deg \iota_3(x) = (\bar{1},\bar{1},\deg x).$$

$\mathbb{Z}_2^5$-grading: Weyl group

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading Γ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$$

$$\deg \iota_3(\mathbf{i}) = a + b + c_1, \ \deg \iota_3(\mathbf{j}) = a + b + c_2, \ \deg \iota_3(\mathbf{l}) = a + b + c_3.$$

Theorem

Let Γ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \oplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{\mu \in \mathrm{Aut}(\mathbb{Z}_2^5) : \mu(T) = T\}.$$

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Remark

Any $\psi \in \mathrm{Stab}(\Gamma)$ fixes E_i and multiplies $\iota_1(1), \iota_2(1), \iota_3(\mathbf{i}), \iota_3(\mathbf{j}), \iota_3(\mathbf{l})$, by either 1 or -1. Hence $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^5$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading

Take an element $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$ and consider the following elements in $\mathbb{A}$:

$$
\begin{aligned}
&E = E_1,\ \widetilde{E} = 1 - E = E_2 + E_3,\\
&\nu(a) = \mathbf{i}\iota_1(a) \quad \text{for all} \quad a \in \mathbb{O}_0,\\
&\nu_\pm(x) = \iota_2(x) \pm \mathbf{i}\iota_3(\bar{x}) \quad \text{for all} \quad x \in \mathbb{O},\\
&S^\pm = E_3 - E_2 \pm \frac{\mathbf{i}}{2}\iota_1(1).
\end{aligned}
$$

$\mathbb{A}$ is then 5-graded:

$$\mathbb{A} = \mathbb{A}_{-2} \oplus \mathbb{A}_{-1} \oplus \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2,$$

with $\mathbb{A}_{\pm 2} = \mathbb{F}S^\pm$, $\mathbb{A}_{\pm 1} = \nu_\pm(\mathbb{O})$, and $\mathbb{A}_0 = \mathbb{F}E \oplus \Big(\mathbb{F}\widetilde{E} \oplus \nu(\mathbb{O}_0)\Big)$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading

The $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ combines with this $\mathbb{Z}$-grading

$$\mathbb{A} = \mathbb{F}S^- \oplus \nu^-(\mathbb{O}) \oplus \mathbb{A}_0 \oplus \nu^+(\mathbb{O}) \oplus \mathbb{F}S^+$$

to give a fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading as follows:

$$\begin{aligned} \deg S^\pm &= (\pm 2, \bar{0}, \bar{0}, \bar{0}), \\ \deg \nu_\pm(x) &= (\pm 1, \deg x), \\ \deg E = 0 &= \deg \widetilde{E}, \\ \deg \nu(a) &= (0, \deg a), \end{aligned}$$

for homogeneous elements $x \in \mathbb{O}$ and $a \in \mathbb{O}_0$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading: Weyl group

Theorem

Let Γ be the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the Albert algebra. Then

$$W(\Gamma) = \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).$$

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Remark

One can show that $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$, which is isomorphic to $\mathbb{F}^\times \times \mathbb{Z}_2^3$.

$\mathbb{Z}_3^3$-grading

Recall that the Okubo algebra can be defined on the octonions, with new multiplication:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

where τ is the order three automorphism of $\mathbb{O}$ given by:

$$\tau(e_i) = e_i,\ i = 1, 2, \quad \tau(u_j) = u_{j+1},\ \tau(v_j) = v_{j+1},\ j = 1, 2, 3.$$

$\mathbb{Z}_3^3$-grading

Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathbb{O}$. Then the multiplication in the Albert algebra

$$\mathbb{A} = \oplus_{i=1}^{3} \big(\mathbb{F}E_i \oplus \tilde{\iota}_i(\mathbb{O})\big)$$

becomes:

$$E_i^{\circ 2} = E_i, \quad E_i \circ E_{i+1} = 0,$$

$$E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2}\tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),$$

$$\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x * y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),$$

for $i = 1, 2, 3$ and $x, y \in \mathbb{O}$.

$\mathbb{Z}_3^3$-grading

Assume now char $\mathbb{F} \neq 3$. Then the $\mathbb{Z}_3^2$-grading on the Okubo algebra is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{O}, *)$:

$$\begin{aligned} \varphi_1(e_1) &= \omega e_1, & \varphi_1(u_1) &= u_1, \\ \varphi_2(e_1) &= e_1, & \varphi_2(u_1) &= \omega u_1, \end{aligned}$$

where ω is a primitive cubic root of unity in $\mathbb{F}$.

$\mathbb{Z}_3^3$-grading

The commuting order 3 automorphisms φ_1, φ_2 of $(\mathbb{O}, *)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_i(\varphi_j(x)).$$

On the other hand, the linear map $\varphi_3 \in \mathrm{End}(\mathcal{A})$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_{i+1}(x),$$

is another order 3 automorphism, which commutes with φ_1 and φ_2.

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The subgroup of $\mathrm{Aut}(\mathbb{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$.

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All the homogeneous components have dimension 1.

$\mathbb{Z}_3^3$-grading: Weyl group

The $\mathbb{Z}_3^3$-grading is determined by

$$\begin{aligned}
\deg\left(\sum_{i=1}^{3} \tilde{\iota}_i(e_1)\right) &= (\bar{1}, \bar{0}, \bar{0}), \\
\deg\left(\sum_{i=1}^{3} \tilde{\iota}_i(u_1)\right) &= (\bar{0}, \bar{1}, \bar{0}), \\
\deg\left(\sum_{i=1}^{3} \omega^{-i} E_i\right) &= (\bar{0}, \bar{0}, \bar{1}),
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\end{aligned}$$

Theorem

Let Γ be the $\mathbb{Z}_3^3$-grading on the Albert algebra. Then $W(\Gamma)$ is the commutator subgroup of $\mathrm{Aut}(\mathbb{Z}_3^3)$*, i.e.,*

$$W(\Gamma) \cong SL_3(3).$$

$\mathbb{Z}_3^3$-grading: Weyl group

Why $SL_3(3)$ and not $GL_3(3)$?

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Consider the $\mathbb{Z}_3^3$-grading Γ^- determined by

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\end{aligned}$$

Then, for $X_1 \in \mathbb{A}_{(\bar{1},\bar{0},\bar{0})}$, $X_2 \in \mathbb{A}_{(\bar{0},\bar{1},\bar{0})}$, $X_3 \in \mathbb{A}_{(\bar{0},\bar{0},\bar{1})}$, we have:

$$(X_1 \circ X_2) \circ X_3 = \begin{cases} \omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma, \\ \omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-. \end{cases}$$

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Hence Γ and Γ^- are equivalent, but NOT isomorphic, gradings.

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Hence Γ and Γ^- are equivalent, but NOT isomorphic, gradings.
Besides, any fine $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ is isomorphic to either Γ or Γ^-, so $W(\Gamma)$ has index two in $\mathrm{Aut}(U(\Gamma)) \cong GL_3(3)$.

$\mathbb{Z}_3^3$-grading and the Tits construction

Let $\mathcal{R} = \mathrm{Mat}_3(\mathbb{F})$. Then

$$\mathbb{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2,$$

with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.

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with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.
The product in $\mathbb{A}$ satisfies $\mathcal{R}_i \circ \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$ (mod 3) and:

$\circ$	a'_0	b'_1	c'_2
a_0	$(a \circ a')_0$	$(\bar{a}b')_1$	$(c'\bar{a})_2$
b_1	$(\bar{a}'b)_1$	$(b \times b')_2$	$(\overline{bc'})_2$
c_2	$(c\bar{a}')_2$	$(\overline{b'c})_0$	$(c \times c')_1$

where

- $a \circ a' = \frac{1}{2}(aa' + a'a)$,
- $a \times b = a \circ b - \frac{1}{2}(\mathrm{tr}(a)b + \mathrm{tr}(b)a) + \frac{1}{2}\big(\mathrm{tr}(a)\,\mathrm{tr}(b) - \mathrm{tr}(ab)\big)1$,
- $\bar{a} = a \times 1 = \frac{1}{2}\big(\mathrm{tr}(a)1 - a\big)$.

$\mathbb{Z}_3^3$-grading and the Tits construction

Assume char $\mathbb{F} \neq 3$. Take Pauli matrices in $\mathcal{R}$:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where ω, ω^2 are the primitive cubic roots of 1, which satisfy

$$x^3 = 1 = y^3, \quad yx = \omega xy.$$

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These Pauli matrices give a grading by $\mathbb{Z}_3^2$ on $\mathcal{R}$, with

$$\mathcal{R}_{(\alpha_1,\alpha_2)} = \mathbb{F}x^{\alpha_1}y^{\alpha_2}.$$

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This grading combines with the $\mathbb{Z}_3$-grading on $\mathbb{A}$ induced by Tits construction, to give the unique, up to equivalence, fine grading by $\mathbb{Z}_3^3$ of the Albert algebra.

$\mathbb{Z}_3^3$-grading and the Tits construction

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq \mathbb{A}.$$

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Then, for any $\alpha, \beta \in \mathbb{Z}_3^3$:

$$Z^\alpha \circ Z^\beta = \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

where

$$\tilde{\psi}(\alpha, \beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3) - (\alpha_1\beta_2 + \alpha_2\beta_1).$$

$\mathbb{Z}_3^3$-grading and the Tits construction

Consider now the elements (Racine 1990, unpublished)

$$W^\alpha := \omega^{-\alpha_1\alpha_2} Z^\alpha.$$

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$$W^\alpha := \omega^{-\alpha_1\alpha_2} Z^\alpha.$$

$$W^\alpha \circ W^\beta = \omega^{-\alpha_1\alpha_2-\beta_1\beta_2} Z^\alpha \circ Z^\beta$$

$$= \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha+\mathbb{Z}_3\beta)\leq 1,\\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{otherwise,}\end{cases}$$

$$= \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)+(\alpha_1\beta_2+\alpha_2\beta_1)} W^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha+\mathbb{Z}_3\beta)\leq 1,\\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)+(\alpha_1\beta_2+\alpha_2\beta_1)} W^{\alpha+\beta} & \text{otherwise.}\end{cases}$$

The Albert algebra as a twisted group algebra

Theorem (Griess 1990)

The Albert algebra is, up to isomorphism, the twisted group algebra

$$\mathbb{A} = \mathbb{F}_\sigma[\mathbb{Z}_3^3],$$

with

$$\sigma(\alpha,\beta) = \begin{cases} \omega^{\psi(\alpha,\beta)} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \le 1, \\ -\frac{1}{2}\omega^{\psi(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

where

$$\psi(\alpha,\beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3).$$

Fine gradings on the Albert algebra

Theorem (Draper–Martín-González 2009 (char = 0), E.–Kochetov 2012)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *The $\mathbb{Z}_2^5$-grading obtained by combining the natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading by $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
4. *The $\mathbb{Z}_3^3$-grading with $\dim \mathbb{A}_g = 1\ \forall g$ (char $\mathbb{F} \neq 3$).*

Fine gradings on the Albert algebra

Theorem (Draper–Martín-González 2009 (char $= 0$), E.–Kochetov 2012)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of* $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$*).*
2. *The* $\mathbb{Z}_2^5$*-grading obtained by combining the natural* $\mathbb{Z}_2^2$*-grading on* 3×3 *hermitian matrices with the fine grading by* $\mathbb{Z}_2^3$ *of* $\mathbb{O}$*.*
3. *The* $\mathbb{Z} \times \mathbb{Z}_2^3$*-grading obtained by combining a* 5*-grading and the* $\mathbb{Z}_2^3$*-grading on* $\mathbb{O}$*.*
4. *The* $\mathbb{Z}_3^3$*-grading with* $\dim \mathbb{A}_g = 1\ \forall g$ *(*$\operatorname{char} \mathbb{F} \neq 3$*).*

All the gradings up to isomorphism on $\mathbb{A}$ have been classified too (E.–Kochetov).

Gradings on F_4

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The adjoint mapt

$$\mathrm{Ad} : \mathbf{Aut}(\mathbb{A}) \to \mathbf{Aut}(\mathfrak{f}_4)$$

(where $\mathbb{A} = H_3(\mathbb{O})$ is the Albert algebra) is an isomorphism.

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Therefore, we can transfer the classification of gradings on the Albert algebra to the simple Lie algebra of type F_4.

Gradings on F_4

Theorem

Up to equivalence, the fine gradings on $\mathfrak{f}_4$ are

- *the Cartan grading,*
- *a grading by $\mathbb{Z}_2^5$, obtained by combining the $\mathbb{Z}_2^2$-grading given by the decomposition $\mathfrak{f}_4 = \mathfrak{d}_4 \oplus \text{natural} \oplus \text{spin} \oplus \overline{\text{spin}}$, with the $\mathbb{Z}_2^3$-grading on the octonions (which is the vector space behind the natural and spin representations of $\mathfrak{d}_4$).*
- *a grading by $\mathbb{Z} \times \mathbb{Z}_2^3$, obtained by looking at $\mathfrak{f}_4$ as the Kantor Lie algebra of a structurable algebra: $\mathfrak{f}_4 = \mathcal{K}(\mathbb{O}, -)$, and combining the natural 5-grading on $\mathcal{K}(\mathbb{O}, -)$ and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
- *a $\mathbb{Z}_3^3$-grading (only if $\operatorname{char} \mathbb{F} \neq 3$), with $(\mathfrak{f}_4)_0 = 0$ and where $(\mathfrak{f}_4)_g \oplus (\mathfrak{f}_4)_{-g}$ is a Cartan subalgebra of $\mathfrak{f}_4$ for any $0 \neq g \in \mathbb{Z}_3^3$.*

Triality

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra.
The linear map

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C}) \\ x &\mapsto \quad \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix} \end{aligned}$$

induces an algebra isomorphism

$$\alpha : \mathfrak{Cl}_{\bar{0}}(\mathcal{C}, n) \to \mathrm{End}_{\mathbb{F}}(\mathcal{C}) \times \mathrm{End}_{\mathbb{F}}(\mathcal{C}).$$

Triality

For any $u \in \mathrm{Spin}(\mathcal{C}, n)$, if $\alpha(u) = (\rho_u^+, \rho_u^-)$, then

$$\chi_u(x * y) = \rho_u^-(x) * \rho_u^+(y)$$

for any $x, y \in \mathcal{C}$. (Here $\chi_u(x) = u \cdot x \cdot u^{-1}$ is the natural representation of $\mathrm{Spin}(\mathcal{C}, n)$ on $\mathcal{C}$.)

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This provides a group isomorphism:

$$\begin{aligned} \mathrm{Spin}(\mathcal{C}, n) &\to \mathrm{Tri}(\mathcal{C}, *, n) \\ u &\mapsto \quad (\chi_u, \rho_u^-, \rho_u^+) \end{aligned}$$

where the triality group is defined by

$$\mathrm{Tri}(\mathcal{C}, *, n) := \{(f_1, f_2, f_3) \in O(\mathcal{C}, n)^3 : \\ f_1(x * y) = f_2(x) * f_3(y) \ \forall x, y \in \mathcal{C}\}.$$

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(This isomorphism can be defined at the level of the corresponding affine group schemes.)

Triality Lie algebra

There is also a *local version* of triality for $\operatorname{char}\mathbb{F} \neq 2$.

Let $(S, *, n)$ be any symmetric composition algebra and consider the corresponding orthogonal Lie algebra:

$$\mathfrak{o}(S,n) = \{d \in \operatorname{End}_{\mathbb{F}}(S) : n\big(d(x),y\big) + n\big(x,d(y)\big) = 0 \ \forall x,y \in S\},$$

and the subalgebra of $\mathfrak{o}(S,n)^3$ (with componentwise multiplication):

$$\mathfrak{tri}(S,*,n) = \{(d_1,d_2,d_3) \in \mathfrak{o}(S,n)^3 : d_3(x*y) = d_1(x)*y + x*d_2(y) \ \forall x,y\}$$

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$$\mathfrak{tri}(S, *, n) = \{(d_1, d_2, d_3) \in \mathfrak{o}(S, n)^3 : d_3(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y\}$$

This is the triality Lie algebra.

The map: $\theta : \mathfrak{tri}(S, *, n) \to \mathfrak{tri}(S, *, n)$, $(d_1, d_2, d_3) \mapsto (d_3, d_1, d_2)$ is an automorphism of order 3, (triality automorphism).

Principle of Local Triality

Theorem (Principle of Local Triality)

Let $(S, *, n)$ be an eight dimensional symmetric composition algebra. Then the projection

$$\begin{aligned} \pi_1 : \mathfrak{tri}(S, *, n) &\longrightarrow \mathfrak{o}(S, n) \\ (d_1, d_2, d_3) &\mapsto \quad d_1, \end{aligned}$$

is an isomorphism of Lie algebras.

Freudenthal's Magic Square

Let $(S, *, n)$ and $(S', \star, n')$ be two symmetric composition algebras. One can construct a Lie algebra as follows:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \Big(\oplus_{i=1}^{3} \iota_i(S \otimes S')\Big),$$

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with bracket given by:

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_1, d_2, d_3), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_1', d_2', d_3'), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$,
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = n'(x', y')\theta^i(t_{x,y}) + n(x, y)\theta'^i(t'_{x',y'})$, for some natural triples $t_{x,y}$ and $t'x', y'$.

Freudenthal's Magic Square

		dim S'			
$\mathfrak{g}(S, S')$		1	2	4	8
dim S	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Cyclic compositions (Springer)

Cyclic compositions (Springer)

Definition

A cyclic composition is a 5-tuple $(V, \mathbb{L}, \rho, *, Q)$ consisting of

- a cubic étale $\mathbb{F}$-algebra $\mathbb{L}$ with an $\mathbb{F}$-automorphism ρ of order 3,
- a free $\mathbb{L}$-module V,
- a quadratic form $Q : V \to \mathbb{L}$ with nondegenerate polar form b_Q,
- an $\mathbb{F}$-bilinear multiplication $* : V \times V \to V$ such that, for any $x, y, z \in V$ and $\ell \in L$:

$$(\ell x) * y = \rho(\ell)(x * y), \quad x * (\ell y) = \rho^2(\ell)(x * y),$$
$$Q(x * y) = \rho(Q(x))\rho^2(Q(y)),$$
$$b_Q(x * y, z) = \rho\big(Q(y * z, x)\big) = \rho^2\big(b_Q(z * x, y)\big).$$

Cyclic compositions

Example

Let $(\mathcal{C}, \star, n)$ be a symmetric composition algebra (over $\mathbb{F}$) and let $\mathbb{L} = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ and $\rho : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_2, \alpha_3, \alpha_1)$.
Then $\big(\mathcal{C} \otimes_{\mathbb{F}} \mathbb{L}, \mathbb{L}, \rho, *, Q\big)$, with $Q = (n, n, n)$ and

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_2 \star y_3, x_3 \star y_1, x_1 \star y_2)$$

for any $x_1, \dots, y_3 \in \mathcal{C}$, is a cyclic composition.

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for any $x_1, \dots, y_3 \in \mathcal{C}$, is a cyclic composition.

In this example, the automorphism group scheme is given by:

$$\mathbf{Aut}_{\mathbb{F}}(V, \mathbb{L}, \rho, *, Q) = \mathbf{Tri}(\mathcal{C}, \star, n) \rtimes \mathbf{A}_3 \cong \mathbf{Spin}(\mathcal{C}, n) \rtimes \mathbf{A}_3.$$

Trialitarian algebras (The Book of Involutions)

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Let $(V, \mathbb{L}, \rho, *, Q)$ be an eight-dimensional cyclic composition.

The associative algebra $E = \mathrm{End}_{\mathbb{L}}(V)$ is endowed with the involution σ determined by Q and an isomorphism

$$\alpha : \mathfrak{Cl}(E, \sigma) \xrightarrow{\sim} {}^{\rho}E \times {}^{\rho^2}E,$$

where the superscripts denote the twist of scalar multiplication (i.e., ${}^{\rho}E$ is E as an $\mathbb{F}$-algebra with involution, but with the new $\mathbb{L}$-module structure defined by $\ell \cdot a = \rho(\ell)a$).

(In the example above, this isomorphism is induced by the isomorphism $\mathfrak{Cl}_{\bar{0}}(\mathcal{C}, n) \simeq \mathrm{End}_{\mathbb{F}}(\mathcal{C}) \times \mathrm{End}_{\mathbb{F}}(\mathcal{C})$.)

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The quadruple $(E, \mathbb{L}, \sigma, \alpha)$ is an example of a trialitarian algebra.

Trialitarian algebras

The subspace

$$\mathcal{L}(E) := \{x \in \mathrm{Skew}(E, \sigma) : \alpha(\text{“}x\text{”}) = (x, x)\}$$

is a central simple Lie algebra of type D_4.

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Theorem

$$\mathbf{Aut}\big(\mathcal{L}(E)\big) \simeq \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha).$$

Remark

Conjugation gives a natural morphism

$$\mathrm{Int} : \mathbf{Aut}(V, \mathbb{L}, \rho, *, Q) \to \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha).$$

Type I, II, III gradings

From now on the ground field $\mathbb{F}$ will be assumed to be algebraically closed of characteristic not two.

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Let $\mathcal{L}$ be the simple Lie algebra of type D_4.

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If $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ is a grading and $\eta : G^D \to \mathbf{Aut}(\mathcal{L})$ the corresponding morphism of group schemes, then the image of $\pi\eta$ is a diagonalizable subgroupscheme of the constant scheme $\mathbf{S}_3$, so it corresponds to an abelian subgroup of the symmetric group S_3, and hence its order is 1, 2 or 3. The grading Γ will be said to have Type I, II, or III respectively.

Type III gradings

- The classification of type I or II gradings follow the same lines as the classification of gradings for D_n, $n \geq 5$.
- Type III gradings do not appear in characteristic 3.

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- Type III gradings do not appear in characteristic 3.

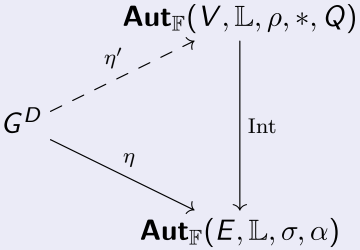
From now on we will deal with type III gradings Γ on $\mathcal{L}$. If $(E, \mathbb{L}, \sigma, \alpha)$ is the trialitarian algebra over $\mathbb{F}$, the isomorphism $\mathbf{Aut}\big(\mathcal{L}(E)\big) \simeq \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha)$ allows us to transfer Γ to a grading on $(E, \mathbb{L}, \sigma, \alpha)$.

Lifting to $\mathbf{Aut}(V, \mathbb{L}, \rho, *, Q)$

Lifting to $\mathbf{Aut}(V, \mathbb{L}, \rho, *, Q)$

Theorem

Any type III grading, identified with a morphism $\eta : G^D \to \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha)$*, can be lifted to a grading on the cyclic composition* $(V, \mathbb{L}, \rho, *, Q)$*:*



Gradings on $(V, \mathbb{L}, \rho, *, Q)$

Gradings on $(V, \mathbb{L}, \rho, *, Q)$

Theorem

Let Γ be a Type III grading by an abelian group G on the eight-dimensional cyclic composition $(V, \mathbb{L}, \rho, *, Q)$ over an algebraically closed field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2, 3$, *and let $\Gamma_{\mathbb{L}}$ be the induced grading on $\mathbb{L}$.*

1. *If $V_e = 0$, then $(V, \mathbb{L}, \rho, *, Q)$ is isomorphic to $(\mathcal{O}, \star, n) \otimes (\mathbb{L}, \rho)$ as a graded cyclic composition algebra, where $(\mathcal{O}, \star, n)$ is the Okubo algebra, endowed with a G-grading $\Gamma_{\mathcal{O}}$ with $\mathcal{O}_e = 0$, and the grading on $(\mathcal{O}, \star, n) \otimes (\mathbb{L}, \rho)$ is $\Gamma_{\mathcal{O}} \otimes \Gamma_{\mathbb{L}}$.*

Gradings on $(V, \mathbb{L}, \rho, *, Q)$

Theorem (continued)

2. *Otherwise,* $(V, \mathbb{L}, \rho, *, Q)$ *is isomorphic to* $(\mathcal{C}, \bullet, n) \otimes (\mathbb{L}, \rho)$ *as a graded cyclic composition algebra, where* $(\mathcal{C}, \bullet, n)$ *is the para-octonion algebra, endowed with a* G*-grading* $\Gamma_{\mathcal{C}}$*, and the grading on* $(\mathcal{C}, \bullet, n) \otimes (\mathbb{L}, \rho)$ *is* $\Gamma_{\mathcal{C}} \otimes \Gamma_{\mathbb{L}}$*.*

Gradings on $(V, \mathbb{L}, \rho, *, Q)$

Theorem (continued)

2. *Otherwise,* $(V, \mathbb{L}, \rho, *, Q)$ *is isomorphic to* $(\mathcal{C}, \bullet, n) \otimes (\mathbb{L}, \rho)$ *as a graded cyclic composition algebra, where* $(\mathcal{C}, \bullet, n)$ *is the para-octonion algebra, endowed with a* G*-grading* $\Gamma_{\mathcal{C}}$*, and the grading on* $(\mathcal{C}, \bullet, n) \otimes (\mathbb{L}, \rho)$ *is* $\Gamma_{\mathcal{C}} \otimes \Gamma_{\mathbb{L}}$*.*

The proof uses the fact that $\mathcal{J}(\mathbb{L}, V) = \mathbb{L} \oplus V$ is the Albert algebra, and we know classification of the gradings on this algebra.

Gradings on D_4

Gradings on D_4

Theorem

Up to equivalence, there are three fine gradings of Type III on the simple Lie algebra of type D_4 over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2, 3$. Their universal groups are $\mathbb{Z}^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3^3$.

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Theorem

Let $\mathbb{F}$ be an algebraically closed field and let $\mathcal{L}$ be the simple Lie algebra of type D_4 over $\mathbb{F}$.

1. *If $\operatorname{char}\mathbb{F} \neq 2, 3$ then there are, up to equivalence, 17 fine gradings on $\mathcal{L}$.*
2. *If $\operatorname{char}\mathbb{F} = 3$ then there are, up to equivalence, 14 fine gradings on $\mathcal{L}$.*

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That's all.
Thanks