Octonions in low characteristics

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Composition algebras

Definition

A composition algebra over a field $\mathbb{F}$ is a triple $(C, \cdot, n)$ where

- C is a vector space over $\mathbb{F}$,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to \mathbb{F}$ is a multiplicative $(n(x \cdot y) = n(x)n(y)\ \forall x, y \in C)$ nonsingular quadratic form.

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The unital composition algebras are called Hurwitz algebras.

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

$$x^{\cdot 2} - n(x,1)x + n(x)1 = 0$$

for any x. $(n(x,y) := n(x+y) - n(x) - n(y).)$

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They are endowed with an involution, the standard conjugation:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

Cayley-Dickson doubling process

Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let λ be a nonzero scalar in the ground field $\mathbb{F}$. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$(a + bu) \cdot (c + du) = (a \cdot c + \lambda \bar{d} \cdot b) + (d \cdot a + b \cdot \bar{c})u,$$
$$n(a + bu) = n(a) - \lambda n(b).$$

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Notation: $CD(A, \mu, \lambda) := CD\big(CD(A, \mu), \lambda\big).$

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following:

(i) *The ground field $\mathbb{F}$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*
If $\operatorname{char} \mathbb{F} \neq 2$, *these are the algebras $CD(\mathbb{F}, \alpha)$.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra (or algebra of octonions) $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

Generalized Hurwitz Theorem

Corollary

Every Hurwitz algebra over a field $\mathbb{F}$ of characteristic $\neq 2$ is obtained by applying the Cayley-Dickson doubling process to $\mathbb{F}$ at most three times.

Split Hurwitz algebras

Proposition

Two Hurwitz algebras are isomorphic if and only if their norms are isometric.

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For each dimension 2, 4 or 8, there is a unique, up to isomorphism, Hurwitz algebra with isotropic norm.

- $\mathbb{F} \times \mathbb{F}$ with $n\big((\alpha, \beta)\big) = \alpha\beta$,
- $\mathrm{Mat}_2(\mathbb{F})$ with $n = \det$,
- $\mathcal{C}_s := CD\big(\mathrm{Mat}_2(\mathbb{F}), 1\big)$ (the *split Cayley algebra*).

Cayley algebras and Malcev algebras

If $\operatorname{char}\mathbb{F} \neq 2$ and $\mathcal{C}$ is a Cayley algebra, then $\mathcal{C} = \mathbb{F}1 \oplus \mathcal{C}_0$, where $\mathcal{C}_0$ is the subspace orthogonal to $\mathbb{F}1$.
For $x, y \in \mathcal{C}_0$, $[x, y] := xy - yx \in \mathcal{C}_0$ and

$$xy = -\frac{1}{2}n(x, y)1 + \frac{1}{2}[x, y].$$

Besides,

$$[[x, y], y] = 2n(x, y)y - 2n(y, y)x,$$

so the multiplication in $\mathcal{C}$ and its norm are determined by the bracket in $\mathcal{C}_0$.

Cayley algebras and Malcev algebras

Theorem (Sagle 1962, Kuzmin 1968, Filippov 1976)

If $\operatorname{char}\mathbb{F} \neq 2, 3$, *the anticommutative algebra* $\mathcal{C}_0$ *is a central simple non-Lie Malcev algebra, and any such algebra is, up to isomorphism, of this form.*

Cayley algebras and simple Lie algebras of type G_2

Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ of type X_r over the complex numbers, and a Chevalley basis $\mathcal{B}$, let $\mathfrak{g}_{\mathbb{Z}}$ be the $\mathbb{Z}$-span of $\mathcal{B}$ (a Lie algebra over $\mathbb{Z}$). The Lie algebra $\mathfrak{g}_{\mathbb{F}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$ is the Chevalley algebra of type X_r.

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Theorem

- *The Chevalley algebra of type G_2 is isomorphic to $\mathfrak{Der}(\mathcal{C}_s)$.*

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Theorem

- *The Chevalley algebra of type G_2 is isomorphic to $\mathfrak{Der}(\mathcal{C}_s)$.*
- *For any Cayley algebra $\mathcal{C}$, the Lie algebra $\mathfrak{Der}(\mathcal{C})$ is a twisted form of the Chevalley algebra $\mathfrak{Der}(\mathcal{C}_s)$.*

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- *For any Cayley algebra $\mathcal{C}$, the Lie algebra $\mathfrak{Der}(\mathcal{C})$ is a twisted form of the Chevalley algebra $\mathfrak{Der}(\mathcal{C}_s)$.*
- *If $\operatorname{char} \mathbb{F} \neq 2, 3$, then $\mathfrak{Der}(\mathcal{C})$ is simple.*

Cayley algebras and simple Lie algebras of type G_2

Theorem (Jacobson 1931, Barnes 1961)

If char $\mathbb{F} \neq 2, 3$,

- *Any twisted form of the Chevalley algebra of type G_2 is isomorphic to $\mathfrak{Der}(\mathcal{C})$ for a Cayley algebra $\mathcal{C}$.*
- *Two Cayley algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic if and only if their Lie algebras of derivations are isomorphic.*

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Theorem (Jacobson 1931, Barnes 1961)

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Sketch of a 'modern' proof

For any Cayley algebra $\mathcal{C}$, the adjoint map

$$\begin{aligned} \mathrm{Ad} : \mathbf{Aut}(\mathcal{C}) &\longrightarrow \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{C})\big) \\ f &\mapsto \mathrm{Ad}(f) : d \mapsto fdf^{-1}, \end{aligned}$$

is an isomorphism of affine group schemes. □

Jacobian

In any algebra $\mathcal{A}$, the Jacobian of the elements x_1, x_2, x_3 is

$$J(x_1, x_2, x_3) := [[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2].$$

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Expand to get:

$$\begin{aligned} J(x_1, x_2, x_3) &= \Big((x_1 x_2 - x_2 x_1)x_3 - x_3(x_1 x_2 - x_2 x_1)\Big) + \text{cyclically} \\ &= \Big((x_1 x_2)x_3 - x_1(x_2 x_3)\Big) + \cdots \\ &= \sum_{\sigma \in \boldsymbol{\Sigma}_3} (-1)^\sigma \big(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\big), \end{aligned}$$

where $(x, y, z) := (xy)z - x(yz)$ is the associator.

Jacobian

But in any Cayley algebra $\mathcal{C}$,

$$x^2 y = x(xy) \quad \text{and} \quad yx^2 = (yx)x$$

for any x, y. That is, $(x, x, y) = 0 = (y, x, x)$, and hence:

$$J(x_1, x_2, x_3) = \sum_{\sigma \in \Sigma_3} (-1)^\sigma \left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right) = 6(x_1, x_2, x_3).$$

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- *$\mathfrak{Der}(\mathcal{C})$ is a semisimple Lie algebra, but not a direct sum of simple ideals.*

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- *$\mathfrak{Der}(\mathcal{C})$ is a semisimple Lie algebra, but not a direct sum of simple ideals.*
- *$\mathfrak{Der}(\mathcal{C})$ contains a unique proper ideal: $\operatorname{ad}(\mathcal{C}_0)$, isomorphic to $\mathcal{C}_0$, and the quotient $\mathfrak{Der}(\mathcal{C})/\operatorname{ad}(\mathcal{C}_0)$ is isomorphic again to $\mathcal{C}_0$.*

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Remark

Actually, there are no 'prime' non-Lie Malcev algebras over fields of characteristic 3.

The adjoint map

In spite of this strange behavior, still we get:

Theorem

Let $\mathcal{C}$ be a Cayley algebra over a field $\mathbb{F}$ of characteristic 3, the adjoint map

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The proof uses the fact that, even in characteristic 3, any derivation of $\mathfrak{Der}(\mathcal{C})$ is inner.

Characteristic 3

Corollary

Denote by Isom(Cayley), Isom(G_2), *and* Isom($\bar{A}_2$), *the sets of isomorphism classes of Cayley algebras, twisted forms of the Chevalley algebra of type* G_2, *and twisted forms of* $\mathfrak{psl}_3(\mathbb{F})$, *respectively.*
Then we have bijections:

$$\begin{array}{ccccc} \mathrm{Isom}(\bar{A}_2) & \longleftrightarrow & \mathrm{Isom}(\mathrm{Cayley}) & \longleftrightarrow & \mathrm{Isom}(G_2) \\ [\mathcal{C}_0] & \leftarrow & [\mathcal{C}] & \rightarrow & [\mathfrak{Der}(\mathcal{C})] \end{array}$$

The Lie algebra of derivations

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Theorem

Let $\mathcal{C}$ be a Cayley algebra over a field $\mathbb{F}$ of characteristic 2, then the Lie algebra $\mathfrak{Der}(\mathcal{C})$ is isomorphic to the projective special linear Lie algebra $\mathfrak{psl}_4(\mathbb{F})$.

The Lie algebra of derivations

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The isomorphism class of $\mathfrak{Der}(\mathcal{C})$ does not depend on $\mathcal{C}$!!

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- Any $d \in \mathfrak{Der}(\mathcal{C})$ preserves $\mathcal{C}_0$ and $\mathbb{F}1$, and $1 \in \mathcal{C}_0$!!!

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- Any $d \in \mathfrak{Der}(\mathcal{C})$ preserves $\mathcal{C}_0$ and $\mathbb{F}1$, and $1 \in \mathcal{C}_0$!!!
- Hence d induces a linear map on the six-dimensional quotient $\mathcal{C}_0/\mathbb{F}1$, which is endowed with a nondegenerate alternating bilinear form induced by the norm n. ($n(x,x) = 2n(x) = 0$!!)

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- Hence d induces a linear map on the six-dimensional quotient $\mathcal{C}_0/\mathbb{F}1$, which is endowed with a nondegenerate alternating bilinear form induced by the norm n. $(n(x,x) = 2n(x) = 0!!)$
- This embeds $\mathfrak{Der}(\mathcal{C})$ into the symplectic Lie algebra $\mathfrak{sp}_6(\mathbb{F})$, and hence into $\mathfrak{sp}_6(\mathbb{F})^{(2)}$, which is isomorphic to $\mathfrak{psl}_4(\mathbb{F})$. But $\dim \mathfrak{psl}_4(\mathbb{F}) = 14 = \dim \mathfrak{Der}(\mathcal{C})$. □

G_2 and $\bar{A}_3$

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Corollary

In characteristic 2, the Chevalley algebra of type G_2 is isomorphic to $\mathfrak{psl}_4(\mathbb{F})$ (the classical simple Lie algebra of type A_3).

$\bar{A}_3$ and C_3

Theorem

Let $\mathbb{F}$ be a field of characteristic 2. Then the affine group scheme of automorphisms of $\mathfrak{psl}_4(\mathbb{F})$ is isomorphic to the affine group scheme of automorphisms of the algebra with involution $(\mathrm{Mat}_6(\mathbb{F}), t_s)$, where t_s is the canonical symplectic involution.

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Sketch of proof

Any automorphism of $\mathrm{Mat}_6(\mathbb{F})$ commuting with the involution t_s restricts to an automorphism of the Lie algebra $\mathfrak{sp}_6(\mathbb{F})^{(2)}$. This induces a closed embedding of group schemes. But the two group schemes involved are connected, smooth and of the same dimension. □

Characteristic 2

Corollary

Let $\mathbb{F}$ be a field of characteristic 2. The map

$$(\mathcal{B}, \tau) \quad \mapsto \quad \mathrm{Skew}(\mathcal{B}, \tau)^{(2)}$$

that sends any central simple associative algebra of degree 6 over $\mathbb{F}$ endowed with a symplectic involution $(\mathcal{B}, \tau)$ to the second derived power of the Lie algebra of its skew-symmetric elements $\mathrm{Skew}(\mathcal{B}, \tau)$, gives a bijection between the set of isomorphism classes of such pairs $(\mathcal{B}, \tau)$ to the set of isomorphism classes of twisted forms over $\mathbb{F}$ of the Lie algebra $\mathfrak{psl}_4(\mathbb{F})$.

A. Castillo-Ramírez and A. Elduque.
Some special features of Cayley algebras, and G_2, in low characteristics.
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Thanks