Order 3 elements in G_2 and idempotents in symmetric composition algebras

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Composition algebras

$$\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{O}$$

Definition

- A **composition algebra** over a field is a triple $(C, \cdot, n)$ where
  - C is a vector space,
  - $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
  - $n : C \to \mathbb{F}$ is a multiplicative ($n(x \cdot y) = n(x)n(y)\ \forall x, y \in C$) nonsingular quadratic form.
- The unital composition algebras are called **Hurwitz algebras**.

Hurwitz algebras

- Hurwitz algebras exist only in dimension 1, 2, 4, or 8.

  (These are too the possible dimensions of the finite-dimensional arbitrary composition algebras.)

- The two-dimensional Hurwitz algebras are just the quadratic étale algebras.

- The four-dimensional Hurwitz algebras are the quaternion algebras.

- The eight-dimensional Hurwitz algebras are termed **octonion (or Cayley) algebras**.

Hurwitz algebras

Theorem

- *Hurwitz algebras are isomorphic iff their norms are isometric.*
- *For each dimension 2, 4, or 8, there is a unique, up to isomorphism, Hurwitz algebra with isotropic norm:*
  - $\mathbb{F} \times \mathbb{F}$ *with* $n\big((\alpha, \beta)\big) = \alpha\beta$,
  - $\mathrm{Mat}_2(\mathbb{F})$ *with* $n = \det$,
  - *The algebra of Zorn matrices (or* **split Cayley algebra***):*

$$\mathcal{C}_s = \left\{ \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{F}, u, v \in \mathbb{F}^3 \right\}, \quad \textit{with}$$

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \cdot \begin{pmatrix} \alpha' & u' \\ v' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + (u \mid v') & \alpha u' + \beta' u - v \times v' \\ \alpha' v + \beta v' + u \times u' & \beta\beta' + (v \mid u') \end{pmatrix},$$

$$n\left(\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \right) = \alpha\beta - (u \mid v).$$

Pseudo-octonions (Okubo 1978)

Let $\mathbb{F}$ be a field of characteristic $\neq 2, 3$ containing a primitive cubic root ω of 1.

On the vector space $\mathfrak{sl}_3(\mathbb{F})$ consider the multiplication:

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

and norm: $$n(x) = -\frac{1}{2} \operatorname{tr}(x^2).$$

Then, for any x, y,

$$n(x * y) = n(x)n(y), \quad (x * y) * x = n(x)y = x * (y * x).$$

In particular, $(\mathfrak{sl}_3(\mathbb{F}), *, n)$ is a *composition algebra*.

Denote by $P_8(\mathbb{F})$ the algebra thus defined (**algebra of pseudo-octonions**).

- $P_8(\mathbb{F})$ makes sense in characteristic 2, because $\mathrm{tr}(x^2)$ 'is a multiple of 2' if $\mathrm{tr}(x) = 0$.
- Okubo and Osborn (1981) gave an 'ad hoc' definition of $P_8(\mathbb{F})$ over fields of characteristic 3 by means of its multiplication table.

Okubo algebras

In order to define Okubo algebras over arbitrary fields consider the Pauli matrices:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

in $\mathrm{Mat}_3(\mathbb{C})$, which satisfy

$$x^3 = y^3 = 1, \quad xy = \omega yx.$$

For $i, j \in \mathbb{Z}/3\mathbb{Z}$, $(i, j) \neq (0, 0)$, define

$$x_{i,j} := \frac{\omega^{ij}}{\omega - \omega^2} x^i y^j.$$

$\{x_{i,j} : (i, j) \neq (0, 0)\}$ is a basis of $\mathfrak{sl}_3(\mathbb{C})$.

Okubo algebras

$$x_{i,j} * x_{k,l} = \omega x_{i,j} x_{k,l} - \omega^2 x_{k,l} x_{i,j} - \frac{\omega - \omega^2}{3} \operatorname{tr}(x_{i,j} x_{k,l}) 1$$
$$= \begin{cases} x_{i+k,j+l} \\ 0 \\ -x_{i+k,j+l} \end{cases} \qquad \bigl(x_{0,0} := 0\bigr)$$

depending on $\begin{vmatrix} i & j \\ k & l \end{vmatrix}$ being equal to 0, 1 or 2 (modulo 3).

Miraculously, the ω's disappear!

Besides, $n(x_{i,j}) = 0$ for any i, j, and

$$n(x_{i,j}, x_{k,l}) = \begin{cases} 1 & \text{for } (i,j) = -(k,l), \\ 0 & \text{otherwise.} \end{cases}$$

Okubo algebras

Thus, the $\mathbb{Z}$-module

$$\mathcal{O}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\,\{x_{i,j} : -1 \leq i,j \leq 1,\ (i,j) \neq (0,0)\}$$

is closed under $*$, and n restricts to a nonsingular multiplicative quadratic form on $\mathcal{O}_{\mathbb{Z}}$.

Definition

Let $\mathbb{F}$ be an arbitrary field. Then

$$\mathcal{O}_{\mathbb{F}} := \mathcal{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F},$$

with the induced multiplication and nonsingular quadratic form, is called the **split Okubo algebra** over $\mathbb{F}$.

The twisted forms of $\mathcal{O}_{\mathbb{F}}$ are called **Okubo algebras** over $\mathbb{F}$.

Symmetric composition algebras

Definition

A composition algebra $(\mathcal{C}, *, n)$ is said to be **symmetric** if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in \mathcal{C}$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in \mathcal{C}$.

Markus Rost, around 1994, realized that this is the right class of algebras to deal with the phenomenon of triality.

Examples

- Okubo algebras are symmetric composition algebras.
- Given any Hurwitz algebra $(\mathcal{B}, \cdot, n)$, the algebra $(\mathcal{B}, \bullet, n)$, where

$$x \bullet y = \bar{x} \cdot \bar{y}$$

  is called the associated **para-Hurwitz** algebra (Okubo-Myung 1980).

  Para-Hurwitz algebras are symmetric.

Theorem (Okubo-Osborn 1981, E.-Pérez-Izquierdo 1996)

Any eight-dimensional symmetric composition algebra is either a para-Hurwitz algebra or an Okubo algebra.

Symmetric composition algebras and triality

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Write

$$L_x(y) = x * y = R_y(x).$$

$$L_x R_x = n(x)\mathrm{id} = R_x L_x \quad \Longrightarrow \quad \begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}^2 = n(x)\mathrm{id}$$

Therefore, the map $x \mapsto \begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}$ extends to an isomorphism of algebras with involution

$$\Phi : \big(\mathfrak{Cl}(\mathcal{C}, n), \tau\big) \longrightarrow \big(\mathrm{End}(\mathcal{C} \oplus \mathcal{C}), \sigma_{n \perp n}\big)$$

Spin group

Consider the *spin group*:

$$\mathrm{Spin}(\mathcal{C}, n) = \left\{ u \in \mathfrak{Cl}(\mathcal{C}, n)^{\times}_{\bar{0}} : u \cdot \mathcal{C} \cdot u^{-1} \subseteq \mathcal{C},\ u \cdot \tau(u) = 1 \right\}.$$

For any $u \in \mathrm{Spin}(\mathcal{C}, n)$,

$$\Phi(u) = \begin{pmatrix} \rho_u^- & 0 \\ 0 & \rho_u^+ \end{pmatrix}$$

for some $\rho_u^\pm \in \mathrm{O}(\mathcal{C}, n)$ such that

$$\chi_u(x * y) = \rho_u^+(x) * \rho_u^-(y)$$

for any $x, y \in \mathcal{C}$, where $\chi_u(x) = u \cdot x \cdot u^{-1}$.

The natural and the two half-spin representations are linked!

Spin group

Theorem

Let $(\mathcal{C}, *, n)$ *be an eight-dimensional symmetric composition algebra. Then:*

$$\begin{aligned}\mathrm{Spin}(\mathcal{C}, n) \simeq \{(f_0, f_1, f_2) \in \mathrm{O}^+(\mathcal{C}, n)^3 :\\ f_0(x * y) = f_1(x) * f_2(y) \quad \forall x, y \in \mathcal{C}\}\end{aligned}$$

$$u \quad \mapsto \quad (\chi_u, \rho_u^+, \rho_u^-)$$

Moreover, the set of related triples *(the set on the right hand side) has cyclic symmetry.*

The cyclic symmetry on the right hand side induces an outer automorphism of order 3 (*trialitarian automorphism*) of $\mathrm{Spin}(\mathcal{C}, n)$. The subgroup of the elements fixed by this automorphism is $\mathrm{Aut}(\mathcal{C}, *, n)$.

Symmetric compositions are either para-Hurwitz or Okubo

Sketch of proof

- If $(\mathcal{C}, *, n)$ is a symmetric composition algebra over $\mathbb{F}$, there is a field extension $\mathbb{K}/\mathbb{F}$ of degree ≤ 3 such that $(\mathcal{C}_{\mathbb{K}}, *, n)$ contains a nonzero idempotent. Hence we may assume that there exists $0 \neq e \in \mathcal{C}$ with $e * e = e$. Then $n(e) = 1$.

- Consider the new multiplication

$$x \cdot y = (e * x) * (y * e).$$

Then $(\mathcal{C}, \cdot, n)$ is a Hurwitz algebra with unity $1 = e$, and the map $\tau : x \mapsto e * (e * x) = n(e, x)e - x * e$ is an automorphism of both $(\mathcal{C}, *, n)$ and of $(\mathcal{C}, \cdot, n)$, such that $\tau^3 = \mathrm{id}$.

- If $\tau = \mathrm{id}$, $(\mathcal{C}, *, n)$ is para-Hurwitz, otherwise it may be either para-Hurwitz or Okubo.

Classification of Okubo algebras

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 3$, containing a primitive cubic root of 1. By restriction we obtain a natural isomorphism

$$\mathbf{PGL}_3 \simeq \mathbf{Aut}\big(\mathrm{Mat}_3(\mathbb{F})\big) \to \mathbf{Aut}\big((\mathfrak{sl}_3(\mathbb{F}), *, n)\big)$$

of affine group schemes.

Theorem (E.-Myung 1991, 1993)

The map

$$\begin{aligned} \left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{central simple degree } 3\\ \textit{associative algebras}\end{array}\right\} &\longrightarrow \left\{\begin{array}{c}\textit{Isomorphism classes}\\ \textit{of Okubo algebras}\end{array}\right\}\\ [\mathcal{A}] &\mapsto [(\mathcal{A}_0, *, n)] \end{aligned}$$

is bijective.

Classification of Okubo algebras

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 3$, not containing primitive cubic roots of 1. Let $\mathbb{K} = \mathbb{F}[X]/(X^2 + X + 1)$.

Theorem (E.-Myung 1991, 1993)

The map

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{pairs } (\mathcal{B},\sigma)\text{, where } \mathcal{B} \text{ is a simple}\\ \text{degree 3 associative algebra}\\ \text{over } \mathbb{K} \text{ and } \sigma \text{ a } \mathbb{K}/\mathbb{F}\text{-involution}\\ \text{of the second kind}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of Okubo algebras}\end{array}\right\}$$

$$[(\mathcal{B},\sigma)] \qquad \mapsto \qquad [(\mathrm{Skew}(\mathcal{B},\sigma)_0, *, n)]$$

is bijective.

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Theorem (Chernousov-E.-Knus-Tignol 2013)

Let $(\mathcal{O}, *, n)$ be the split Okubo algebra over a field $\mathbb{F}$ (char $\mathbb{F} = 3$).

- **Aut**$(\mathcal{O}, *, n)$ *is not smooth:* $\dim \mathbf{Aut}(\mathcal{O}, *, n) = 8$ *while* $\mathfrak{Der}(\mathcal{O}, *, n)$ *is a simple (nonclassical) Lie algebra of dimension* 10.
- $\mathbf{Aut}(\mathcal{O}, *, n) = \mathbf{HD}$, *where* $\mathbf{H} = \mathbf{Aut}(\mathcal{O}, *, n)_{\text{red}} = \mathbf{Aut}(\mathcal{O}, *, n, e)$, *where e is the* **quaternionic idempotent** *and* $\mathbf{D} \simeq \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$.
- *The map*

$$H^1(\mathbb{F}, \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \to H^1\big(\mathbb{F}, \mathbf{Aut}(\mathcal{O}, *, n)\big)$$

induced by the inclusion $\mathbf{D} \hookrightarrow \mathbf{Aut}(\mathcal{O}, *, n)$, *is surjective.*

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Recall that $\mathcal{O}$ is spanned by elements $x_{i,j}$, $(i, j) \neq (0, 0)$ (indices modulo 3). It is actually generated by $x_{1,0}$ and $x_{0,1}$. Given $0 \neq \alpha, \beta \in \mathbb{F}$, the elements

$$x_{1,0} \otimes \alpha^{\frac{1}{3}},\ x_{0,1} \otimes \beta^{\frac{1}{3}} \in \mathcal{O} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$$

generates, by multiplication and linear combinations over $\mathbb{F}$, a twisted form of $(\mathcal{O}, *, n)$. Denote it by $\mathcal{O}_{\alpha,\beta}$.

Theorem (E. 1997)

Any Okubo algebra over $\mathbb{F}$ (char $\mathbb{F} = 3$) is isomorphic to $\mathcal{O}_{\alpha,\beta}$ for some $0 \neq \alpha, \beta \in \mathbb{F}$.

Idempotents and order 3 automorphisms

Let $(\mathcal{C}, *, n)$ be a symmetric composition algebra, $0 \neq e = e * e$.

- With $x \cdot y = (e * x) * (y * e)$, $(\mathcal{C}, \cdot, n)$ is a Hurwitz algebra with unity $1 = e$.
- $\tau : x \mapsto e * (e * x)$ is an automorphism of both $(\mathcal{C}, \cdot, n)$ and $(\mathcal{C}, *, n)$, and $\tau^3 = \mathrm{id}$.
- If $\tau = \mathrm{id}$, then $(\mathcal{C}, *, n)$ is para-Hurwitz and e its *para-unit* $(e * x = x * e = \bar{x} := n(e, x)e - x \ \forall x)$.
- $x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y})$, so $(\mathcal{C}, *, n)$ is recovered from the Hurwitz algebra $(\mathcal{C}, \cdot, n)$ and τ.[1]

[1] Conversely, if τ is an order 3 automorphism of a Hurwitz algebra $(\mathcal{C}, \cdot, n)$, then the *Petersson algebra* $(\mathcal{C}, *, n)$, where $x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y})$, is a symmetric composition algebra, and $e = 1$ is an idempotent.

Idempotents and order 3 automorphisms

Therefore,

Order 3 automorphisms of Cayley algebras

$$\Updownarrow$$

Idempotents in symmetric composition algebras

Idempotents and order 3 automorphisms

Moreover,

- The subalgebra of fixed points by τ:

$$\mathrm{Fix}(\tau) := \{x \in \mathcal{C} : \tau(x) = x\}$$

  coincides with the centralizer of e:

$$\mathrm{Cent}_{(\mathcal{C},*,n)}(e) := \{x \in \mathcal{C} : e * x = x * e\}.$$

- The centralizer in the group scheme of automorphisms of $(\mathcal{C}, \cdot, n)$ of τ coincides with the stabilizer of e in the group scheme of automorphisms of $(\mathcal{C}, *, n)$:

$$\mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau) = \mathbf{Stab}_{\mathbf{Aut}(\mathcal{C},*,n)}(e).$$

Order 3 automorphisms

Let τ be an order 3 automorphism of a Cayley algebra $(\mathcal{C}, \cdot, n)$ over a field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 3$.

There are two different possibilities:

- There is an element $w \in \mathcal{C} \setminus \mathbb{F}1$ with $w^{\cdot 2} + w + 1 = 0$ such that for any $x \in \mathcal{C}$

$$\tau(x) = w \cdot x \cdot w^2.$$

  In this case, the subalgebra $\mathrm{Fix}(\tau)$ of the elements fixed by τ is $\mathcal{K} = \mathbb{F}1 + \mathbb{F}w$ (a quadratic étale subalgebra).

- The subalgebra $\mathrm{Fix}(\tau)$ is a quaternion subalgebra of $\mathcal{C}$ containing an element $w \in \mathcal{C} \setminus \mathbb{F}1$ such that $w^{\cdot 2} + w + 1 = 0$.

$\dim \mathrm{Fix}(\tau) = 2$

- These automorphisms correspond to the idempotents of a para-Cayley algebra, different from its para-unit.
- Any two such automorphisms are conjugate in $\mathrm{Aut}(\mathcal{C}, \cdot, n)$.
- $\mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau) = \mathbf{Stab}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(w)$, and this is, up to isomorphism, the special unitary group $\mathbf{SU}(\mathcal{W}, \sigma)$, for $\mathcal{W} = \mathcal{K}^{\perp}$ and a suitable hermitian form σ.

$\dim \mathrm{Fix}(\tau) = 4$

- These automorphisms correspond to the idempotents of an Okubo algebra.
- Any two such automorphisms are conjugate in $\mathrm{Aut}(\mathcal{C}, \cdot, n)$ if and only if the corresponding quaternion subalgebras are isomorphic.
  In particular, if $\mathbb{F}$ contains the cubic roots of 1, then $\mathcal{C}$ is the split Cayley algebra, the corresponding symmetric composition algebra is the split Okubo algebra, and any two such automorphisms are conjugate.
- $\mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau)$ is isomorphic to $\mathbf{SL}_1(\mathcal{Q}) \times \mathbf{SL}_1(\mathcal{K})/\boldsymbol{\mu}_2$, where $\mathcal{Q} = \mathrm{Fix}(\tau)$.

Idempotents in para-Cayley algebras

Let $(\mathcal{C}, \cdot, n)$ be a Cayley algebra and $(\mathcal{C}, \bullet, n)$ the associated para-Cayley algebra ($x \bullet y = \bar{x} \cdot \bar{y}$ for any $x, y \in \mathcal{C}$). Then

- If either $\operatorname{char} \mathbb{F} \neq 3$ and $(\mathcal{C}, \cdot, n)$ contains a subalgebra isomorphic to the quadratic étale algebra $\mathcal{K} = \mathbb{F}[X]/(X^2 + X + 1)$, or $\operatorname{char} \mathbb{F} = 3$ and $(\mathcal{C}, \cdot, n)$ is split, then $(\mathcal{C}, \bullet, n)$ contains the para-unit and a unique conjugacy class of other idempotents:

  $$\{\text{idempotents of } (\mathcal{C}, \bullet, n)\} = \{1\} \cup \{w \in \mathcal{C} \setminus \mathbb{F}1 : w^{\cdot 2} + w + 1 = 0\}.$$

  All the idempotents, with the exception of the para-unit 1, are conjugate under $\operatorname{Aut}(\mathcal{C}, \cdot, n) = \operatorname{Aut}(\mathcal{C}, \bullet, n)$.
- Otherwise, the only idempotent of $(\mathcal{C}, \bullet, n)$ is its para-unit.

Idempotents in Okubo algebras

- An Okubo algebra $(\mathcal{O}, *, n)$ over a field $\mathbb{F}$ of characteristic $\neq 3$ contains an idempotent if and only if it contains a subalgebra isomorphic to the para-Hurwitz algebra $\overline{\mathcal{K}}$ associated to $\mathcal{K}$.

  In this case, the centralizer of any idempotent e is a para-quaternion subalgebra containing a subalgebra isomorphic to $\overline{\mathcal{K}}$, and two idempotents are conjugate if and only if the corresponding quaternion algebras are isomorphic.

  In particular, if $\mathbb{F}$ contains the cubic roots of 1, then $(\mathcal{O}, *, n)$ contains a unique conjugacy class of idempotents.

- $(\mathcal{O}, *, n)$ is split if and only if n is isotropic and there is an idempotent.

Idempotents of Okubo algebras

Definition

An idempotent of an Okubo algebra $(\mathcal{O}, *, n)$ ($\mathrm{char}\,\mathbb{F} = 3$) is said to be:

- **quaternionic**, if its centralizer contains a para-quaternion algebra,
- **quadratic**, if its centralizer contains a para-quadratic algebra and no para-quaternion subalgebra,
- **singular**, otherwise.

Proposition (Chernousov-E.-Knus-Tignol 2013)

Among Okubo algebras, only the split one contains a quaternionic idempotent, and only one.

Order 3 automorphisms

Let $(\mathcal{C}, \cdot, n)$ be a Cayley algebra over a field $\mathbb{F}$ of characteristic 3, and let τ be an order 3 automorphism of $(\mathcal{C}, \cdot, n)$. Then $(\mathcal{C}, \cdot, n)$ is the split Cayley algebra.

Consider the root space decomposition of the Lie algebra of derivations relative to a Cartan subalgebra:

$$\mathcal{L} = \mathfrak{der}(\mathcal{C}, \cdot, n) = \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha \right).$$

There are four different possibilities:

Order 3 automorphisms I

I) $(\tau - \mathrm{id})^2 = 0$.

- In this case τ is conjugate to $\exp(\delta)$, with $0 \neq \delta \in \mathcal{L}_\alpha$, for α a long root.
- Up to conjugation, there is only one such automorphism.
- These automorphisms correspond to the quaternionic idempotent of the split Okubo algebra $(\mathcal{O}, *, n)$.
- $\mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau)$ is the derived subgroup of the parabolic subgroup corresponding to a short root.
  Up to isomorphism, this is the group $\mathbf{Aut}(\mathcal{O}, *, n)_{\mathrm{red}}$.

Order 3 automorphisms II

II) $(\tau - \mathrm{id})^2 \neq 0$ and $\mathrm{Fix}(\tau)$ contains a 2-dimensional Hurwitz subalgebra $\mathcal{K}$.

- If $\mathbb{F}$ is algebraically closed, all these automorphisms are conjugate.
- There is a short exact sequence

$$1 \longrightarrow \left(\mathbf{K} \rtimes \boldsymbol{\mu}_{3,[\mathcal{K}]}\right) \times \mathbf{N} \longrightarrow \mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau) \longrightarrow \mathsf{C}_2 \longrightarrow 1$$

  where $\mathbf{K}$ and $\mathbf{N}$ are isomorphic to $\mathbf{G}_a^2$, and $\boldsymbol{\mu}_{3,[\mathcal{K}]}$ is a twisted form of $\boldsymbol{\mu}_3$.
- These automorphisms correspond to the quadratic idempotents of Okubo algebras.

Order 3 automorphisms III

III) τ is conjugate to $\exp(\delta)$, with $\delta \in \mathcal{L}_\alpha$, α short.

- All these automorphisms are conjugate.
- $\mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau)$ is the derived subgroup of the parabolic subgroup corresponding to a long root.
- These automorphisms correspond to the idempotents of the split para-Cayley algebra other than its para-unit.

Order 3 automorphisms IV

IV) τ is the composition of automorphisms in I) and III) corresponding to orthogonal roots.

- All these automorphisms are conjugate.
- $\mathbf{Cent}_{\mathbf{Aut}(\mathcal{C},\cdot,n)}(\tau)$ is the unipotent radical of a standard Borel subgroup.
- These automorphisms correspond to the singular idempotents of the split Okubo algebra.

Idempotents in Okubo algebras

Let $(\mathcal{O}, *, n)$ be an Okubo algebra over a field $\mathbb{F}$ ($\operatorname{char}\mathbb{F} = 3$). Then the map

$$\begin{aligned} g : \mathcal{O} &\longrightarrow \mathbb{F} \\ x &\mapsto n(x, x * x) \end{aligned}$$

satisfies

$$g(x + y) = g(x) + g(y), \qquad g(\alpha x) = \alpha^3 g(x),$$

for $x, y \in \mathcal{O}$, $\alpha \in \mathbb{F}$.

It turns out that $\dim_{\mathbb{F}^3} g(\mathcal{O})$ is 1, 3 or 8.

Idempotents in Okubo algebras

Theorem

1. *If* $\dim_{\mathbb{F}^3} g(\mathcal{O}) = 8$, *then* $(\mathcal{O}, *, n)$ *contains no idempotents.*
2. *If* $\dim_{\mathbb{F}^3} g(\mathcal{O}) = 3$, $(\mathcal{O}, *, n)$ *contains a unique quadratic idempotent for each class* $[\mathcal{K}, a]$ *with* $\mathcal{K}$ *a quadratic étale algebra,* $a \in \mathcal{K} \setminus \mathcal{K}^{\cdot 3}$, $n(a) = 1$, *and* $g(\mathcal{O}) = \mathbb{F}^3(a + \bar{a})$.
3. *If* $(\mathcal{O}, *, n)$ *is the split Okubo algebra, then it contains:*
   - 3.1 *a unique quaternionic idempotent,*
   - 3.2 *a conjugacy class of quadratic idempotents for each isomorphism class of quadratic étale algebras,*
   - 3.3 *a unique conjugacy class of singular idempotents.*

$[\mathcal{K}, a] = [\mathcal{K}', a']$ iff there is an isomorphism $\varphi : \mathcal{K} \to \mathcal{K}'$ such that $\varphi(a) \in (\mathcal{K}')^3 \cdot a'$.

Thank you!