# Triality, composition algebras, and gradings on $D_4$

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Dedicated to Efim Zelmanov

# Gradings

$G$ abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

$G$-grading on $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

**Cartan grading:**

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$$

(root space decomposition of a semisimple complex Lie algebra).

# Examples

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This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank} \mathfrak{g}$.

# Examples

**Pauli matrices:** $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

($\epsilon$ a primitive $n$th root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

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This grading induces a grading on $\mathfrak{sl}_n(\mathbb{F})$.

# Fine gradings

$$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g'\in G'} \mathcal{A}'_{g'}, \quad \text{gradings on } \mathcal{A}.$$

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- $\Gamma$ is a refinement of $\Gamma'$ if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
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**Remark**

Any grading is a coarsening of a fine grading.

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where

$$G^D : \mathrm{Alg}_{\mathbb{F}} \longrightarrow \mathrm{Grp}$$
$$R \mapsto G^D(R) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R) \left(\simeq \mathrm{Hom}_{\mathrm{Grp}}(G, R^\times)\right),$$

$$\mathbf{Aut}(\mathcal{A}) : \mathrm{Alg}_{\mathbb{F}} \longrightarrow \mathrm{Grp}$$
$$R \mapsto \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes_{\mathbb{F}} R).$$

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where

$$\eta_R(f)\big(x_g \otimes r\big) = x_g \otimes f(g)r$$

for $f \in G^D(R) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R)$, $x_g \in \mathcal{A}_g$ and $r \in R$.

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Conversely,

$$\eta : G^D \to \mathbf{Aut}(\mathcal{A}) \quad \Longrightarrow \quad \eta_{\mathbb{F}G}(\mathrm{id}) \in \mathrm{Aut}(\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G)$$

$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ with

$$\mathcal{A}_g = \{a \in A : \eta_{\mathbb{F}G}(\mathrm{id})(a \otimes 1) = a \otimes g\} \quad \forall g \in G.$$

# Consequences

Given a morphism $\mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$, any grading on $\mathcal{A}$ induces a grading on $\mathcal{B}$.

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**Example**

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If $\mathbf{Aut}(\mathcal{A}) \cong \mathbf{Aut}(\mathcal{B})$, the problems of classifying fine gradings on $\mathcal{A}$ and on $\mathcal{B}$ up to equivalence (or the problem of classifying gradings up to isomorphism) are equivalent.

# Classical Lie algebras

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Assume the ground field is algebraically closed of characteristic not two.

- $B_n, C_n$ $(n \geq 2)$, $D_n$ $(n \geq 5)$:

$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(M_r(\mathbb{F}), \text{involution}).$$

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What about $D_4$?

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There are, up to equivalence, two fine gradings on the octonions (E. 1998):

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{O})$.
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The induced $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type $G_2$ satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in \mathbb{Z}_2^3$.

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There are, up to equivalence, four fine gradings on the Albert algebra –Draper-Martín (char $\mathbb{F} = 0$, 2009); E.-Kochetov (2012)–:

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{A})$.
- A $\mathbb{Z} \times \mathbb{Z}_2^3$-grading related to the fine $\mathbb{Z}_2^3$-grading on the octonions
- A $\mathbb{Z}_2^5$-grading obtained by combining a natural $\mathbb{Z}_2^2$-grading on $3 \times 3$ hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.
- A $\mathbb{Z}_3^3$-grading with $\dim \mathbb{A}_g = 1\ \forall g$ (char $\mathbb{F} \neq 3$).

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The induced $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type $F_4$ satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.

# Composition algebras

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**Definition**

A composition algebra is a triple $(\mathcal{C}, *, n)$, where

- $(\mathcal{C}, *)$ is a (not necessarily associative) algebra,
- $n : C \to \mathbb{F}$ is a nonsingular *multiplicative* quadratic form.

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For Hurwitz algebras, the map $x \mapsto \bar{x} = b_n(x, 1)1 - x$ is an involution such that $x\bar{x} = \bar{x}x = n(x)1$ for any $x$.

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A composition algebra $(\mathcal{C}, *, n)$ is said to be symmetric if its norm is *associative*:

$$b_n(x * y, z) = b_n(x, y * z)$$

for any $x, y, z \in \mathcal{C}$.

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## Example

For any Hurwitz algebra $(\mathcal{C}, *, n)$, its *para-Hurwitz* counterpart is $(\mathcal{C}, \bullet, n)$, with

$$x \bullet y = \bar{x} * \bar{y}$$

for any $x, y \in \mathcal{C}$. These are symmetric composition algebras.

# Okubo algebras

**Example**

Let $\omega \in \mathbb{F}$ be a primitive cube root of unity, then $\mathfrak{sl}_3(\mathbb{F})$, with

- multiplication: $x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)$,
- norm: $n(x) = -\frac{1}{2} \operatorname{tr}(x^2)$,

is a symmetric composition algebra.
Its forms are called Okubo algebras.
(Okubo algebras need a different definition in characteristic three.)

# Symmetric composition algebras

**Theorem**

*With a few exceptions in dimension 2, any symmetric composition algebra is either a para-Hurwitz algebra or an Okubo algebra.*

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Two para-Hurwitz algebras are isomorphic if and only if so are their Hurwitz counterparts.

In characteristic not three, the classification of Okubo algebras, up to isomorphism, is given in terms of central simple associative algebras of degree three.
In characteristic three it follows a different path.

# Triality

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra.
The linear map

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C}) \\ x &\mapsto \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix} \end{aligned}$$

induces an algebra isomorphism between the Clifford algebra of $(\mathcal{C}, n)$ and $\mathrm{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C})$, that restricts to an isomorphism

$$\alpha : \mathfrak{Cl}_{\bar{0}}(\mathcal{C}, n) \to \mathrm{End}_{\mathbb{F}}(\mathcal{C}) \times \mathrm{End}_{\mathbb{F}}(\mathcal{C}).$$

# Triality

For any $u \in \mathrm{Spin}(\mathcal{C}, n)$, if $\alpha(u) = (\rho_u^+, \rho_u^-)$, then

$$\chi_u(x * y) = \rho_u^-(x) * \rho_u^+(y)$$

for any $x, y \in \mathcal{C}$. (Here $\chi_u(x) = u \cdot x \cdot u^{-1}$ is the natural representation of $\mathrm{Spin}(\mathcal{C}, n)$ on $\mathcal{C}$.)

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This provides a group isomorphism:

$$\begin{aligned} \mathrm{Spin}(\mathcal{C}, n) &\to \mathrm{Tri}(\mathcal{C}, *, n) \\ u &\mapsto \quad (\chi_u, \rho_u^-, \rho_u^+) \end{aligned}$$

where the *triality group* is defined by

$$\begin{aligned} \mathrm{Tri}(\mathcal{C}, *, n) := \{&(f_1, f_2, f_3) \in O(\mathcal{C}, n)^3 : \\ &f_1(x * y) = f_2(x) * f_3(y)\ \forall x, y \in \mathcal{C}\}. \end{aligned}$$

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(This isomorphism can be defined at the level of the corresponding affine group schemes.)

# Cyclic compositions (Springer)

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## Definition

A cyclic composition is a 5-tuple $(V, \mathbb{L}, \rho, *, Q)$ consisting of

- a cubic étale $\mathbb{F}$-algebra $\mathbb{L}$ with an $\mathbb{F}$-automorphism $\rho$ of order 3,
- a free $\mathbb{L}$-module $V$ of rank 8,
- a quadratic form $Q : V \to \mathbb{L}$ with nondegenerate polar form $b_Q$,
- an $\mathbb{F}$-bilinear multiplication $* : V \times V \to V$ such that, for any $x, y, z \in V$ and $\ell \in L$:

$$(\ell x) * y = \rho(\ell)(x * y), \quad x * (\ell y) = \rho^2(\ell)(x * y),$$
$$Q(x * y) = \rho(Q(x))\rho^2(Q(y)),$$
$$b_Q(x * y, z) = \rho\big(b_Q(y * z, x)\big) = \rho^2\big(b_Q(z * x, y)\big).$$

# Cyclic compositions

**Example**

Let $(\mathcal{C}, \star, n)$ be a symmetric composition algebra (over $\mathbb{F}$) and let $\mathbb{L} = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ and $\rho : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_2, \alpha_3, \alpha_1)$.
Then $\big(\mathcal{C} \otimes_{\mathbb{F}} \mathbb{L}, \mathbb{L}, \rho, *, Q\big)$, with $Q = (n, n, n)$ and

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_2 \star y_3, x_3 \star y_1, x_1 \star y_2)$$

for any $x_1, \ldots, y_3 \in \mathcal{C}$, is a cyclic composition.

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In this example, the automorphism group scheme is given by:

$$\mathbf{Aut}_{\mathbb{F}}(V, \mathbb{L}, \rho, *, Q) = \mathbf{Tri}(\mathcal{C}, \star, n) \rtimes \mathbf{A}_3 \cong \mathbf{Spin}(\mathcal{C}, n) \rtimes \mathbf{A}_3.$$

# Trialitarian algebras (The Book of Involutions)

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Let $(V, \mathbb{L}, \rho, *, Q)$ be a cyclic composition.

The associative algebra $E = \mathrm{End}_{\mathbb{L}}(V)$ is endowed with the involution $\sigma$ determined by $Q$ and an isomorphism

$$\alpha : \mathfrak{Cl}(E, \sigma) \xrightarrow{\sim} {}^{\rho}E \times {}^{\rho^2}E,$$

where the superscripts denote the twist of scalar multiplication (i.e., ${}^{\rho}E$ is $E$ as an $\mathbb{F}$-algebra with involution, but with the new $\mathbb{L}$-module structure defined by $\ell \cdot a = \rho(\ell)a$).

(In the example above, this isomorphism is induced by the isomorphism $\mathfrak{Cl}_{\bar{0}}(\mathcal{C}, n) \simeq \mathrm{End}_{\mathbb{F}}(\mathcal{C}) \times \mathrm{End}_{\mathbb{F}}(\mathcal{C})$.)

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The quadruple $(E, \mathbb{L}, \sigma, \alpha)$ is an example of a trialitarian algebra.

# Trialitarian algebras

The subspace

$$\mathcal{L}(E) := \{x \in \mathrm{Skew}(E,\sigma) : \alpha(\text{“}x\text{”}) = (x,x)\}$$

is a central simple Lie algebra of type $D_4$.

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**Theorem**

$$\mathbf{Aut}\big(\mathcal{L}(E)\big) \simeq \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha).$$

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**Theorem**

$$\mathbf{Aut}\big(\mathcal{L}(E)\big) \simeq \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha).$$

**Remark**

Conjugation gives a natural morphism

$$\mathrm{Int} : \mathbf{Aut}(V, \mathbb{L}, \rho, *, Q) \to \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha).$$

# Type I, II, III gradings

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If $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ is a grading and $\eta : G^D \to \mathbf{Aut}(\mathcal{L})$ the corresponding morphism of group schemes, then the image of $\pi\eta$ is a diagonalizable subgroupscheme of the constant scheme $\mathbf{S}_3$, so it corresponds to an abelian subgroup of the symmetric group $S_3$, and hence its order is 1, 2 or 3. The grading $\Gamma$ will be said to have Type I, II, or III respectively.

# Type III gradings

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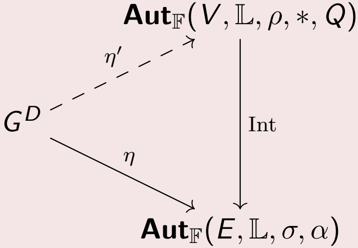
From now on we will deal with type III gradings $\Gamma$ on $\mathcal{L}$. If $(E, \mathbb{L}, \sigma, \alpha)$ is the trialitarian algebra over $\mathbb{F}$, the isomorphism $\mathbf{Aut}\big(\mathcal{L}(E)\big) \simeq \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha)$ allows us to transfer $\Gamma$ to a grading on $(E, \mathbb{L}, \sigma, \alpha)$.

# Lifting to **Aut**$(V, \mathbb{L}, \rho, *, Q)$

# Lifting to $\mathbf{Aut}(V, \mathbb{L}, \rho, *, Q)$

**Theorem**

*Any type III grading, identified with a morphism* $\eta : G^D \to \mathbf{Aut}(E, \mathbb{L}, \sigma, \alpha)$*, can be lifted to a grading on the cyclic composition* $(V, \mathbb{L}, \rho, *, Q)$*:*



# Gradings on $(V, \mathbb{L}, \rho, *, Q)$

# Gradings on $(V, \mathbb{L}, \rho, *, Q)$

**Theorem**

*Let $\Gamma$ be a Type III grading by an abelian group $G$ on the cyclic composition $(V, \mathbb{L}, \rho, *, Q)$ over an algebraically closed field $\mathbb{F}$,* char $\mathbb{F} \neq 2, 3$, *and let $\Gamma_{\mathbb{L}}$ be the induced grading on $\mathbb{L}$.*

1. *If $V_e = 0$, then $(V, \mathbb{L}, \rho, *, Q)$ is isomorphic to $(\mathcal{O}, \star, n) \otimes (\mathbb{L}, \rho)$ as a graded cyclic composition algebra, where $(\mathcal{O}, \star, n)$ is the Okubo algebra, endowed with a $G$-grading $\Gamma_{\mathcal{O}}$ with $\mathcal{O}_e = 0$, and the grading on $(\mathcal{O}, \star, n) \otimes (\mathbb{L}, \rho)$ is $\Gamma_{\mathcal{O}} \otimes \Gamma_{\mathbb{L}}$.*
2. *Otherwise, $(V, \mathbb{L}, \rho, *, Q)$ is isomorphic to $(\mathcal{C}, \bullet, n) \otimes (\mathbb{L}, \rho)$ as a graded cyclic composition algebra, where $(\mathcal{C}, \bullet, n)$ is the para-Cayley algebra, endowed with a $G$-grading $\Gamma_{\mathcal{C}}$, and the grading on $(\mathcal{C}, \bullet, n) \otimes (\mathbb{L}, \rho)$ is $\Gamma_{\mathcal{C}} \otimes \Gamma_{\mathbb{L}}$.*

# Gradings on $(V, \mathbb{L}, \rho, *, Q)$

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The proof uses the fact that $\mathcal{J}(\mathbb{L}, V) = \mathbb{L} \oplus V$ is the Albert algebra, and there is a classification of the gradings on this algebra.

# Gradings on $D_4$

# Gradings on $D_4$

**Theorem**

*Up to equivalence, there are three fine gradings of Type III on the simple Lie algebra of type $D_4$ over an algebraically closed field $\mathbb{F}$,* char $\mathbb{F} \neq 2, 3$. *Their universal groups are $\mathbb{Z}^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3^3$.*

# Gradings on $D_4$

**Theorem**

*Up to equivalence, there are three fine gradings of Type III on the simple Lie algebra of type $D_4$ over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2, 3$. Their universal groups are $\mathbb{Z}^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3^3$.*

**Theorem**

*Let $\mathbb{F}$ be an algebraically closed field and let $\mathcal{L}$ be the simple Lie algebra of type $D_4$ over $\mathbb{F}$.*

1. *If $\operatorname{char}\mathbb{F} \neq 2, 3$ then there are, up to equivalence, 17 fine gradings on $\mathcal{L}$.*
2. *If $\operatorname{char}\mathbb{F} = 3$ then there are, up to equivalence, 14 fine gradings on $\mathcal{L}$.*

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That's all.
Thanks