nonassociative algebra and its applications

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Invariant connections on symmetric spaces

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ABSTRACT: A classical problem in Differential Geometry, the determination of the invariant affine connections in the simply connected irreducible symmetric spaces, is equivalent to the algebraic problem of computing the set $\mathrm{Hom}_S(T \otimes_{\mathbb{R}} T, T)$ for any $\mathbb{Z}_2$-graded simple Lie algebra $L = S \oplus T$. The algebraic problem is solved using known information about the Lie triple system structure on T, because the simple $\mathbb{Z}_2$-graded Lie algebra $L = S \oplus T$ is just the embedding for the simple Lie triple system T. It turns out that the set of homomorphisms contains non trivial elements if and only if T is related to a simple Jordan algebra. Now it is possible to come back to the geometric context to describe the affine connections and express the holonomy and torsion and curvature tensors in algebraic terms.

1. INTRODUCTION.

In another paper of these proceedings [1], it is explained how the geometrical problem of describing the invariant affine connections on a reductive homogeneous space $M = G/H$ is equivalent to the algebraic problem of describing the multiplications $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ with $(\mathrm{Ad}\, h)\alpha(x, y) = \alpha\big((\mathrm{Ad}\, h)(x), (\mathrm{Ad}\, h)(y)\big)$ for any $h \in H$ and any $x, y \in \mathfrak{m}$, where $\mathfrak{m}$ is a vector subspace of the Lie algebra $\mathfrak{g}$ of G such that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$

(where $\mathfrak{h}$ is the Lie subalgebra of $\mathfrak{g}$ which corresponds to the closed subgroup H

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of G) and $AdH(\mathfrak{m}) \subseteq \mathfrak{m}$, a condition equivalent to $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ if H is connected. Such decomposition is called a *reductive decomposition*. This interplay between Geometry and Algebra was proved by Nomizu in [2].

The aim of this paper is to study the connection algebras $(\mathfrak{m}, \alpha)$ that appear in the irreducible symmetric spaces, a special type of reductive homogeneous spaces.

If M is a manifold and ∇ an affine connection, (M, ∇) is a *symmetric space* if for each point $p \in M$ there exists a central symmetry S_p with center at p. Any symmetric space is isomorphic to a space of the form G/H, where the closed subgroup $H \subseteq G$ is such that ${G_0}^\sigma \subseteq H \subseteq G^\sigma$, with σ an involutive automorphism of the Lie group G, G^σ the subgroup of fixed points under σ and ${G_0}^\sigma$ the connected component of the identity in G^σ.

For any symmetric space G/H, there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, because $d\sigma$ provides a $\mathbb{Z}_2$-gradation ($\mathfrak{h}$ and $\mathfrak{g}$, as before, are the Lie algebras of H and G respectively). This kind of decomposition is called *a symmetric decomposition*. The converse is true.

For the simply connected irreducible symmetric spaces $M = G/H$, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is graded-simple. Therefore the geometric problem of computing invariant affine connections in a simply connected irreducible symmetric space $M = G/H$ is equivalent, after applying Nomizu's result, to the algebraic problem of computing the set $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a graded-simple $\mathbb{Z}_2$-graded Lie algebra, because each homomorphism of $\mathfrak{h}$-modules $\tilde{\alpha} \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$ determines an $\mathbb{R}$-bilinear multiplication $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that $\mathrm{ad}\, \mathfrak{h}|_{\mathfrak{m}} \subseteq Der(\mathfrak{m}, \alpha)$ and, since H is connected, this is equivalent to $(\mathrm{Ad}\, h)\alpha(x, y) = \alpha\big((\mathrm{Ad}\, h)(x), (\mathrm{Ad}\, h)(y)\big)$ for any $h \in H$ and any $x, y \in \mathfrak{m}$; that is, it determines a connection algebra and so an invariant affine connection.

In the next section, we will present briefly the solution of the general problem of determining the set $\mathrm{Hom}_S(T \otimes_F T, T)$ for any simple finite-dimensional $\mathbb{Z}_2$-graded Lie algebra $L = S \oplus T$ over an arbitrary field F of characteristic zero. In particular, in the real case we will have obtained an algebraic description of all invariant affine connections in irreducible symmetric spaces (this was done previously in the Riemannian case, using different methods, by Laquer [3,4]). This description will be used in section 3 to determine the holonomy algebra and give explicit formulas for the torsion and curvature tensors associated to each connection.

2. THE SOLUTION OF THE ALGEBRAIC PROBLEM.

The determination of the set $\mathrm{Hom}_S(T \otimes_F T, T)$ for any $\mathbb{Z}_2$-graded simple Lie algebra $L = S \oplus T$ has been done in [5], the purpose of this section is to present the main result in [5, Theorem 4.3] explaining briefly how the structure of a Lie triple system can be effectively used to compute those sets without decomposing $T \otimes_F T$.

From now on all the algebras and systems considered will be assumed to be finite-dimensional over a field F of characteristic 0.

2.1. Basic definitions.

A Lie triple system (L.t.s) is a vector space T endowed with a trilinear product $[x, y, z]$ satisfying:

i) $[x, x, y] = 0$,

ii) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,

iii) $[x, y, -] \in \mathrm{Der}\, T$,

for any $x, y, z \in T$. The set $S = \mathrm{span}\,\langle [x, y, -] \mid x, y \in T\rangle$ is a Lie subalgebra of linear mappings on T and the $\mathbb{Z}_2$-graded Lie algebra $L(T) = S \oplus T$, whose bracket operation is given by:

$$[s_1, s_2] := s_1 s_2 \in S \text{ (the multiplication in } S)$$
$$[s, t] := s(t) \in T \text{ (the natural action)}$$
$$[t_1, t_2] := [t_1, t_2, -] \in S$$

for any $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$, is called the *standard embedding* of T. Note that S and T are the even and odd part of the embedding and T is a faithful S-module. Thus any Lie triple system is nothing else but the odd part of a $\mathbb{Z}_2$-graded Lie algebra with product $[x, y, z] = [[x, y], z]$.

Among the Lie triple systems we are interested in the so called simple ones (without nontrivial ideals). They are just the odd part of simple $\mathbb{Z}_2$-graded Lie algebras (see [6]). Consequently the problem now is to compute the set $\mathrm{Hom}_S(T \otimes_F T, T)$ for any simple Lie triple system T with standard embedding $L(T) = S \oplus T$.

2.2. Examples.

It is not difficult to get elementary examples of simple L.t.s. We point out three of them which are specially relevant for our purposes. First we note that if T is a (simple) L.t.s. and $\mu \in F \setminus \{0\} = F^*$, the vector space T with the product $[x, y, z]^\mu = \mu[x, y, z]$ is a (simple) L.t.s. (in case $\mu \in F^2$ both systems are isomorphic).

Example 1. Any simple Lie algebra A is a simple L.t.s under the trilinear product $[x, y, z] = [[x, y], z]$ ($[\,,\,]$ the Lie bracket in A). In this case $S = \mathrm{ad}\, A = \mathrm{Der}\, A \simeq A$ and the standard embedding is $L(A) = \mathrm{Der}\, A \oplus A \simeq A \oplus A$. Moreover

A is the adjoint module for S and therefore the Lie bracket in A is a nonzero skew-symmetric element in $\mathrm{Hom}_S(A\otimes A, A)$. For any $\mu \in F^*$, we shall refer to the system $(A, [\,,\,,]^\mu)$ as a L.t.s *of adjoint type*.

Over algebraically closed fields, it is known ([7,8]) that

$$\dim \mathrm{Hom}_A(A \otimes A, A) = \begin{cases} 2 & \text{if } A \text{ is of type } A_n \ (n \geq 2), \\ 1 & \text{otherwise.} \end{cases}$$

Consequently, $\mathrm{Hom}_S(A\otimes A, A)$ is spanned by the "Lie bracket in A" for types other than A_n $(n \geq 2)$. But, what happens in the remaining cases? The answer can be found in the following example.

Example 2. For any simple Jordan algebra J of degree $n \geq 2$ with generic trace t, the set J_0 of trace zero elements in J becomes a simple L.t.s via $[x, y, z] = (xz)y - x(zy)$ (the associator of x, z and y). Since $[x, y, -] = [R_y, R_x]$ $(R_x u = ux)$, it follows that $S = [R_J, R_J] = \mathrm{Der}\, J$ and the standard embedding for this system is $L(J_0) = \mathrm{Der}\, J \oplus J_0 \simeq \mathrm{Der}\, J \oplus R_{J_0}$. In this case we can obtain a symmetric element in $\mathrm{Hom}_S(J_0 \otimes J_0, J_0)$ by means of

$$\begin{aligned} J_0 \times J_0 &\longrightarrow J_0 \\ x \otimes y &\mapsto x \cdot y = xy - \frac{1}{n} t(xy)1 \end{aligned}$$

(that is, the projection of the element xy onto J_0) and this product is nontrivial if and only if $n \geq 3$. The systems $(J_0, [\,,\,,]^\mu)$ will be said to be of *Jordan type*.

In case F is algebraically closed, if J is a simple Jordan algebra of degree $n \geq 3$ over F, then either:

i) J is the algebra of symmetric $n \times n$ matrices. In this case $\mathrm{Der}\, J$ is isomorphic to the set of skew-symmetric matrices ([9, Theorem VI.9]) and $L(J_0) \simeq sl(n, F)$.

ii) J is the algebra of symmetric $2n \times 2n$ matrices with respect to the standard symplectic involution. Again in this case $\mathrm{Der}\, J$ is isomorphic to the skew-symmetric matrices with respect to this involution [9, Theorem VI.9] and $L(J_0) \simeq sl(2n, F)$.

iii) J is the exceptional simple Jordan algebra, $\mathrm{Der}\, J$ is a Lie algebra of type F_4 and $L(J_0)$ is a Lie algebra of type E_6 [9, Section IV.11].

In all these cases, $\dim \mathrm{Hom}_S(J_0 \otimes J_0, J_0) \geq 1$, and the main Theorem will assert that, in fact, it is 1.

iv) $J = Mat_n(F)^+$. In this case $J_0 = sl(n, F)$ and $\mathrm{Der}\, J = \mathrm{ad}\, J_0$. Then the triple system J_0 is both of adjoint type (A_{n-1}) and of Jordan type.
Therefore, $\mathrm{Hom}_S(T \otimes_F T, T)$ is spanned by the "Lie bracket in S" and the "Jordan product $\cdot$" for the adjoint type A_n $(n \geq 2)$. This is the answer to the question in Example 1.

The situation in iv) can be settled in a more general context over arbitrary fields of characteristic 0, as the following and final example ([9, Section V.7]) shows:

Example 3. Let (A, j) be a central simple associative involutorial algebra of degree $n \geq 3$ with j an involution of second kind and center $P = F[q]$, a quadratic extension of the base field or isomorphic to $F \times F$. Then the Jordan algebra $J = H(A,j)^+$ of j-symmetric elements of A is central simple of type A and degree n (that is, after extension by scalars it becomes a total $n \times n$ matrix algebra) and the derived algebra $S(A,j)_0 = [S(A,j), S(A,j)]$ of the Lie algebra $S(A,j)^-$ of j-skew elements of A is a central simple Lie algebra of type A_{n-1} [10, Theorem X.8] (Note that if $P = F \oplus F$, $A = B \oplus B^{op}$, with B a central simple associative algebra and j the exchange involution: $j(b_1, b_2) = (b_2, b_1)$. So we can identify $H(A,j)^+ \simeq B^+$ and $S(A,j)^- \simeq B^-$). The converse is also true: any central simple Jordan algebra of type A and degree $n \geq 3$ can be obtained in this manner.

If we consider the L.t.s. of Jordan type $T = J_0 = H(A,j)_0 = \{x \in H(A,j) \mid t(x) = 0\}$ (t the generic trace of $H(A,j)$), we have that $S = \text{Der}\, J = \text{ad}\, S(A,j)_0|_{H(A,j)_0}$ [9].

As the linear map $x \mapsto qx$ of J_0 onto $S(A,j)_0$ is an isomorphism of $S(A,j)_0$-modules, our L.t.s. is both of adjoint type and of Jordan type, and the bracket in $S(A,j)_0$ under this isomorphism provides the product

$$x * y = q[x,y]\,,$$

which is a nonzero skew-symmetric element in $\text{Hom}_{\text{Der}\, J}(J_0 \otimes J_0, J_0)$. Consequently the dimension of this space is at least 2. We shall refer to the systems $(H(A,j)_0, [\,,\,,\,]^\mu)$ as L.t.s. of *adjoint-Jordan type*.

2.3. The centroid of a Lie triple system.

A natural and useful tool in the study of simple algebraic structures is the centroid. Given any L.t.s. T we define the centroid of T in the natural way as $\Gamma = \{\alpha \in End_F(T) \mid \alpha([x,y,z]) = [\alpha(x), y, z] = [x, \alpha(y), z] = [x, y, \alpha(z)] \quad \forall x, y, z \in T\}$. If T is simple, Γ is a field contained in the centroid of $L(T)$, and T over Γ remains simple by scalars extensions.

A technical result in [5, Lemma 4.1] shows that for any simple Lie triple system T with centroid Γ, in order to determine $\text{Hom}_S(T \otimes_F T, T)$ it is enough to determine $\text{Hom}_S(T \otimes_\Gamma T, T)$.

In this way, our problem is reduced to the special case of central simple L.t.s. for which we can extend scalars up to an algebraically closed field. Then, the problem becomes to compute the set of homomorphisms for simple L.t.s. over algebraically closed fields.

Now we shall sketch how to solve this situation.

2.4. Solution in the algebraically closed case.

In this paragraph let T be a simple L.t.s. over an algebraically closed field F and $L(T) = S \oplus T$ be its standard embedding.

The complete classification of such systems was obtained by Lister in 1952 [6]. In 1980 Faulkner [11] gave a method based on the use of the affine Dynkin diagrams (see [12, Ch.8]) that provides an useful and quicker classification. This one will be used here to deduce our results.

From [6,11] the algebra $L(T)$ is a simple ungraded algebra unless T is of adjoint type, and S is a reductive algebra with $\dim Z(S) \leq 1$, consequently the derived subalgebra $[S,S]$ of S is semisimple. In case $Z(S) = 0$, T is an irreducible selfdual module for S; otherwise $T = T_1 \oplus T_2$, T_i being irreducible dual S-modules.

The central idea in Faulkner's classification is to associate a diagram to each simple L.t.s. T, starting with the Dynkin diagram of the semisimple algebra $[S,S]$ and adding a node for each $[S,S]$-irreducible component of T (which represents its minimum weight). All the nodes in the diagrams are equipped by numerical labels representing the linear dependence of the roots of $[S,S]$ and the weights of T involved in the diagrams. These weighted diagrams contain the whole information on the Lie algebra S and the S-module T. They can be grouped into four types.

i) Diagrams representing simple L.t.s. of adjoint type. For these ones, the problem was already solved.

ii) Diagrams representing non irreducible simple L.t.s.. Given such a system T, the center of S is one-dimensional and $T = T_1 \oplus T_2$ with T_i S-irreducible. In this case, there exists a nonzero central element $z \in S$ with $\operatorname{ad} z|_{T_1} = 1$ and $\operatorname{ad} z|_{T_2} = -1$. Therefore $T \otimes T$ is the sum of the eigenspaces 2, 0, -2 and so $\operatorname{Hom}_S(T \otimes T, T) = 0$.

iii) Diagrams representing irreducible simple L.t.s with trivial 0-weight space, and

iv) Diagrams representing irreducible simple L.t.s. with nontrivial 0-weight space.

In the two latter cases the corresponding L.t.s. are selfdual modules and the determination of the set $\operatorname{Hom}_S(T \otimes T, T)$ can be reduced to the problem of bounding the dimension of a certain subspace in the 0-weight space of T (which is 0 for systems of type iii)) because of the following result [5, Lemma 3.3] and [8]:

LEMMA *Let S be a semisimple Lie algebra, H a Cartan subalgebra of S, $T = V(\lambda)$ an irreducible selfdual module with highest weight λ relative to H, v_λ and $v_{-\lambda}$ nonzero weight vectors for λ and $-\lambda$. For any root α relative to H let S_α be the corresponding root space. If T_0 is the 0-weight space in T, the linear map*

$$\begin{aligned} \phi : \operatorname{Hom}_S(T \otimes T, T) &\longrightarrow T_0^\lambda = \{v \in T_0 \mid S_\alpha \cdot v = 0 \quad \forall \alpha \perp \lambda\} \\ \varphi &\mapsto \varphi(v_\lambda \otimes v_{-\lambda}) \end{aligned}$$

is well defined and one-to-one. That is, $\dim \mathrm{Hom}_S(T \otimes T, T) \leq \dim T_0^\lambda$.

So, given a simple L.t.s. T of type iii), from the previous Lemma we get that $\mathrm{Hom}_S(T \otimes T, T) = 0$; and given a T of type iv), by using the technical Lemma in [5, Lemma 3.4] and the information provided by the affine diagrams of types i), iii) and iv) it is possible to obtain that the space $\mathrm{Hom}_S(T \otimes T, T)$ is trivial or one-dimensional, turning out that $\mathrm{Hom}_S(T \otimes T, T) = 0$ unless T is a simple L.t.s. of Jordan type (the L.t.s. i), ii), iii) described in Example 2), in this case $\mathrm{Hom}_S(T \otimes T, T)$ is spanned by the product $\cdot$ (J is not of type A).

2.5. The main Theorem.

Now, taking into account the considerations above, we can establish the main result in [5, Theorem 4.3]:

THEOREM *Let T be a simple Lie triple system over a field F of characteristic zero with centroid Γ and let $L = S \oplus T$ be its standard embedding. Then* $\mathrm{Hom}_S(T \otimes_F T, T) = 0$ *unless either:*

a) *T is of adjoint type with S being a central simple Lie algebra over Γ of type different from A_n $(n \geq 2)$. In this case* $\mathrm{Hom}_S(T \otimes_F T, T) \simeq \mathrm{Hom}_S(S \otimes_F S, S)$, *which is spanned over Γ by the Lie multiplication in S, or*

b) *there exists a central simple Jordan algebra J of degree $n \geq 3$ over Γ and a nonzero scalar $\mu \in \Gamma$ (which can be taken modulo Γ^2) such that T is the Lie triple system $(J_0, [\,,\,,\,]^\mu)$. In this case either:*

   i) *J is not of type A over Γ, then* $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_F J_0, J_0)$ *is spanned over Γ by the product $x \cdot y$,*

   ii) *$J = H(A, j)$ for some central simple associative involutorial algebra (A, j) of second kind over Γ. Let $P = \Gamma[q]$ $(0 \neq q^2 \in \Gamma)$ be the center of A. Then* $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_F J_0, J_0) = \mathrm{Hom}_{S(A,j)_0}(H(A,j)_0 \otimes_F H(A,j)_0, H(A,j)_0)$ *is spanned over Γ by the products $\cdot$ and $*$.*

From the point of view of nonassociative algebras, this result points out the central role played by the multiplication $\cdot$, defined in the set J_0 of the trace zero elements in a Jordan algebra J, in the computation of the sets $\mathrm{Hom}_S(T \otimes T, T)$.

The restriction of the Theorem to the real field gives us the invariant affine connections on the irreducible symmetric spaces. So, in a sense, we can say that the Jordan algebras are responsible for the existence of noncanonical connections on the symmetric spaces.

3. APPLICATIONS TO DIFFERENTIAL GEOMETRY.

In this section we are going to compute some important geometric objects in a symmetric space, namely, the torsion and curvature tensors and the holonomy algebra associated to the invariant affine connections on it.

First of all, we will get expressions for these objects in any reductive homogeneous space in terms of the multiplication in the associated connection algebra. Afterwards we will specialize the previous computations to the symmetric spaces, because for these spaces we know the concrete connection algebras (thanks to the central Theorem in Section 2).

The torsion and curvature tensors are the most important tensors associated to an affine connection ∇, and they are given by the following formulas:

- Torsion tensor: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- Curvature tensor: $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Now, if $M = G/H$ is a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and ∇ is a G-invariant affine connection, let $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be the associated multiplication by Nomizu's result. In this case the torsion and curvature tensors can be expressed in terms of the multiplication α by means of:

$$T(X,Y) = \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}}$$
$$R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_{\mathfrak{m}}, Z) - [[X,Y]_{\mathfrak{h}}, Z]$$

for any $X, Y, Z \in \mathfrak{m}$, where $Z_{\mathfrak{m}}$ and $Z_{\mathfrak{h}}$ denote the projections of $Z \in \mathfrak{g}$ onto $\mathfrak{m}$ and $\mathfrak{h}$ respectively, and we are identifying $\mathfrak{m}$ with the tangent space $T_{\bar{e}}M$.

If M is symmetric, the projection over $\mathfrak{m}$ of the bracket of two elements in $\mathfrak{m}$ is zero, hence:

$$T(X,Y) = \alpha(X,Y) - \alpha(Y,X)$$
$$R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - [[X,Y]_{\mathfrak{h}}, Z].$$

On the other hand, the holonomy algebra $\operatorname{Hol}\nabla$ is the smallest Lie subalgebra of $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$ containing $R(X,Y)$ for all $X, Y \in \mathfrak{m}$ and closed under commutation by ∇_X for any $X \in \mathfrak{m}$.

The previous facts lead us to introduce the following definitions:

Given a Lie triple system T with standard embedding $L(T) = S \oplus T$ and given $\alpha \in \operatorname{Hom}_S(T \otimes_F T, T)$, we define:

i) Torsion tensor: $T^\alpha(x,y) = \alpha(x,y) - \alpha(y,x)$

ii) Curvature tensor: $R^\alpha(x,y)z = \alpha(x,\alpha(y,z)) - \alpha(y,\alpha(x,z)) - [x,y,z]$

iii) Holonomy algebra: $\operatorname{Hol}^\alpha(T)$ = the smallest subalgebra of $\operatorname{End}_F T$ containing $R^\alpha(x,y)$ for all $x, y \in T$ and closed under commutation by the operators $\alpha(x,-)$.

We are going to compute all these concepts for any simple Lie triple system T over an arbitrary field F of characteristic zero. In the particular case $F = \mathbb{R}$, each $\alpha \in \mathrm{Hom}_S(T \otimes_{\mathbb{R}} T, T)$ corresponds to an invariant affine connection in the symmetric space so that T^α, R^α and Hol^α are the true torsion tensor, curvature tensor and holonomy algebra associated to the connection.

According to the main theorem in Section 2 we can assume that T is a central simple Lie triple system over F with standard embedding $L(T) = S \oplus T$ and the pair (T, α) has one of the following forms:

I) **Trivial case:** $\alpha = 0$

Clearly we have for any $x, y \in T$:

$$\begin{cases} T^\alpha(x,y) = 0 \\ R^\alpha(x,y) = -[x,y,-] \\ \mathrm{Hol}^\alpha(T) = \mathrm{alg}\langle [x,y,-] \mid x,y \in T\rangle = S \end{cases}$$

II) **Adjoint type:** there exists a central simple Lie algebra S and $\mu, \eta \in F^* = F \setminus \{0\}$ such that $T = S$ and for any $x, y \in T$:

$$\begin{cases} [x,y,-] = \mu\,\mathrm{ad}[x,y] \\ \alpha(x,y) = \eta[x,y] \end{cases}$$

where $[\,,]$ is the product in S.

Immediately we obtain in this case

$$\begin{cases} T^\alpha(x,y) = 2\eta[x,y] \\ R^\alpha(x,y) = (\eta^2 - \mu)\,\mathrm{ad}[x,y] \\ \mathrm{Hol}^\alpha(T) = \begin{cases} 0 & \text{if } \mu = \eta^2 \\ \mathrm{ad}\, S \simeq S & \text{if } \mu \neq \eta^2 \end{cases} \end{cases}$$

III) **Jordan type:** there exists a central simple Jordan algebra J of degree $n \geq 3$ with generic trace t and $\mu, \eta \in F^*$ such that $T = J_0 = \{x \in J \mid t(x) = 0\}$ is the set of trace zero elements and for any $x, y \in J_0$:

$$\begin{cases} [x,y,-] = \mu[R_x, R_y]|_{J_0} \\ \alpha(x,y) = \eta\, x \cdot y \end{cases}$$

where $x \cdot y = xy - \frac{1}{n}t(xy)1$ and xy is the product in J.

In order to compute the holonomy we need a previous result:

LEMMA *Let J be a central simple Jordan algebra of degree $n \geq 3$ with generic trace t and $J_0 = \{x \in J \mid t(x) = 0\}$ in which we consider the product $x \cdot y = xy - \frac{1}{n}t(xy)1$ and the trace form $t(x,y) = t(xy)$.*

Then the derivation algebra and the Lie multiplication algebra of $(J_0, \cdot)$ are the Lie algebras given by:

$$\mathrm{Der}(J_0, \cdot) = \{d|_{J_0} : d \in \mathrm{Der}\, J\} \quad (\simeq \mathrm{Der}\, J)$$
$$\mathrm{Lie}(J_0, \cdot) = sl(J_0)$$

Furthermore, $(J_0,\cdot)$ is simple.

Proof: If $(a,b,c)=(ab)c-a(bc)=[R_c,R_a](b)$ is the associator in J and $(a,b,c)^{\cdot}=(a\cdot b)\cdot c-a\cdot(b\cdot c)$ the associator in $(J_0,\cdot)$, it is straightforward to check that

$$(a,b,c)^{\cdot}=(a,b,c)-\frac{1}{n}(t(a,b)c-t(b,c)a) \qquad (*)$$

for any $a,b,c\in J_0$. Therefore for any $x,y\in J_0$

$$(x\cdot x,y,x)^{\cdot}=\frac{1}{n}(t(x,y)x\cdot x-t(x\cdot x,y)x).$$

For any $d\in \mathrm{Der}(J_0,\cdot)$, its action on this equality gives

$$\big(t(dx,y)+t(x,dy)\big)\,x\cdot x-\big(t(d(x\cdot x),y)+t(x\cdot x,dy)\big)\,x=0$$

for any $x,y\in J_0$. Since the degree of J is ≥ 3, the set of elements $x\in J_0$ such that x and $x\cdot x$ are linearly independent is a dense subset in the Zariski topology, we obtain from the above that

$$t(dx,y)+t(x,dy)=0$$

for any $x,y\in J_0$. That is, $\mathrm{Der}(J_0,\cdot)$ is contained in the orthogonal Lie algebra related to t: $so(J_0,t)$. Now, extending d to J by means of $d1=0$, we get immediately that $d\in \mathrm{Der}\,J$. Conversely, any derivation of J restricts to a derivation of $(J_0,\cdot)$, so that

$$\mathrm{Der}(J_0,\cdot)=\{d|_{J_0}:d\in \mathrm{Der}\,J\}\simeq \mathrm{Der}\,J\,.$$

It is known that J_0 is an irreducible $\mathrm{Der}\,J$-module, therefore J_0 is an irreducible $\mathrm{Der}(J_0,\cdot)$-module, which implies that $(J_0,\cdot)$ is simple and their derivations are inner ([13, Theorem 3.4] and [14, Lemma 1]), that is, $\mathrm{Der}(J_0,\cdot)\subseteq \mathrm{Lie}(J_0,\cdot)$.

We denote by $R^{\cdot}_x$ the multiplication operator in $(J_0,\cdot)$ given by $R^{\cdot}_x(y)=x\cdot y$. Since $trace(R_x)$ is a multiple of $t(x)$, we have $trace(R^{\cdot}_x)=trace(R_x)=0$ if $x\in J_0$, hence $\mathrm{Lie}(J_0,\cdot)\subseteq sl(J_0)$.

In order to prove the converse, we note that (*) is equivalent to:

$$[R_x,R_y]=[R^{\cdot}_x,R^{\cdot}_y]-\frac{1}{n}\big(t(x,-)y-t(y,-)x\big)$$

for any $x,y\in J_0$. Hence

$$\frac{1}{n}\big(t(x,-)y-t(y,-)x\big)=$$
$$=[R^{\cdot}_x,R^{\cdot}_y]-[R_x,R_y]\in \mathrm{Lie}(J_0,\cdot)+\mathrm{Der}(J_0,\cdot)\subseteq \mathrm{Lie}(J_0,\cdot)$$

since $[R_x,R_y]$ is always a derivation of J. But the linear maps on the left hand side above span the orthogonal Lie algebra $so(J_0,t)$, so that $so(J_0,t)\subseteq \mathrm{Lie}(J_0,\cdot)$. Now we consider the symmetric elements relative to the trace: $H = \{\varphi \in End_F(J_0) \mid trace(\varphi) = 0,\ t(\varphi x, y) = t(x, \varphi y) \quad \forall x, y \in J_0\}$. Then $sl(J_0) = H \oplus so(J_0, t)$, and H is an irreducible module for $so(J_0, t)$. As the operator $R_x^{\cdot}$ is symmetric with respect to the trace form since

$$t(x \cdot y, z) = t((xy)z) - \frac{1}{n}t(xy)t(z) = t((xy)z) = t(y(xz)) = t(y, x \cdot z),$$

it follows that $H \cap \mathrm{Lie}(J_0, \cdot) \neq 0$ and, by irreducibility, that $H \subseteq \mathrm{Lie}(J_0, \cdot)$ and we conclude that $\mathrm{Lie}(J_0, \cdot) = sl(J_0)$. ∎

We now proceed to the computations in this case. It is clear that $T^\alpha \equiv 0$. As regards curvature, we see at once that

$$\begin{aligned} R^\alpha(x, y) &= \eta^2[R_x^{\cdot}, R_y^{\cdot}] - \mu[R_x, R_y] \\ &= (\eta^2 - \mu)[R_x, R_y] + \frac{\eta^2}{n}(t(x, -)y - t(y, -)x). \end{aligned}$$

As for the holonomy algebra, we notice that

$$R^\alpha(x, y) = (\eta^2 - \mu)[R_x, R_y] + \frac{\eta^2}{n}(t(x, -)y - t(y, -)x) \in \mathrm{Der}\, J + so(J_0, t),$$

which is contained in $\mathrm{Lie}(J_0, \cdot)$ and shows that $\mathrm{Hol}^\alpha(J_0) \subseteq \mathrm{Lie}(J_0, \cdot)$, because $\mathrm{Lie}(J_0, \cdot)$ is obviously closed under commutation by $\alpha(x, -) = \eta R_x^{\cdot}$. Besides, $\mathrm{Hol}^\alpha(J_0)$ is closed under commutation by $R_x^{\cdot}$ and these elements generate the Lie algebra $\mathrm{Lie}(J_0, \cdot)$, which yields that $\mathrm{Hol}^\alpha(J_0)$ is an ideal of $\mathrm{Lie}(J_0, \cdot) = sl(J_0)$; but this is a simple algebra, consequently $\mathrm{Hol}^\alpha(J_0)$ is 0 or $sl(J_0)$. As the degree of J is greater than 2, there exists $x \in J_0$ such that x and $x \cdot x = y$ are linearly independent, and for them we have $[R_x, R_y] = 0$ (Jordan identity) and so $R^\alpha(x, y) \neq 0$, therefore $\mathrm{Hol}^\alpha(J_0) = sl(J_0)$. Summarizing, we have obtained

$$\begin{cases} T^\alpha(x, y) = 0 \\ R^\alpha(x, y) = (\eta^2 - \mu)[R_x, R_y] + \frac{\eta^2}{n}(t(x, -)y - t(y, -)x) \\ \mathrm{Hol}^\alpha(T) = sl(J_0) \end{cases}$$

IV) **Adjoint-Jordan type**: there exists a simple associative algebra of degree $n \geq 3$ with an involution of second kind (A, j) and center $K = F[q]$ (quadratic extension of F or isomorphic to $F \times F$) where $q^2 \in F^*$, and $\mu, \eta, \nu \in F^*$ such that $T = H(A, j)_0 = \{x \in A \mid j(x) = x, t(x) = 0\}$ (t the generic trace of $H(A, j)$) and for any $x, y \in T$:

$$\begin{cases} [x, y, -] = \mu\, \mathrm{ad}[x, y]|_T \\ \alpha(x, y) = \eta\, x \cdot y + \nu q\, [x, y] \end{cases}$$

where $x \cdot y = \frac{1}{2}(xy + yx) - \frac{1}{n}t(xy)1$ with xy the product in A, and $[x, y] = xy - yx$ the bracket in A.

In this case first we will find out the derivation algebra and the Lie multiplication algebra, as in the Jordan case. As $\operatorname{ad} x$ is a derivation for both the Lie bracket and the Jordan product in A, it is clear that $\operatorname{ad} qx \in \operatorname{Der}(T, \alpha)$ for all $x \in T$. The converse is true, because

$$\operatorname{Der}(T, \alpha) \subseteq \operatorname{Der}((T, \alpha)^+) = \operatorname{Der}(H(A, J)_0, \cdot) = \operatorname{Der} H(A, j)$$

by the Lemma, where $(T, \alpha)^+$ denotes the algebra defined on T with multiplication given by $\alpha(x, y) + \alpha(y, x)$, and it is known [9, Theorem VI.9] that $\operatorname{Der} H(A, j) = \operatorname{ad} S(A, j)_0|_{H(A,j)_0}$, where $S(A, j) = \{x \in A \mid j(x) = -x\} = qH(A, j)$ and $S(A, j)_0$ its subset of trace zero elements . Therefore

$$\operatorname{Der}(T, \alpha) = \operatorname{ad} S(A, j)_0|_{H(A,j)_0} \simeq S(A, j)_0$$

since $S(A, j)_0$ is a simple Lie algebra. Hence $T = H(A, J)_0 = qS(A, j)_0$ is an irreducible $\operatorname{ad} S(A, j)_0 = \operatorname{Der} T$-module, consequently (T, α) is again a simple algebra and its derivations are inner. As for the Lie multiplication algebra, $\alpha(x, -)$ and $\alpha(-, x)$ are linear combinations of $R^{\cdot}_x$ (with zero trace, by the Lemma) and $\operatorname{ad} qx$, whence $\operatorname{Lie}(T, \alpha) \subseteq sl(T)$. But $\operatorname{Lie}(T, \alpha) \supseteq \operatorname{Lie}((T, \alpha)^+) = \operatorname{Lie}(H(A, j)_0, \cdot)$ and we saw in the Lemma above that $\operatorname{Lie}(H(A, j)_0, \cdot) = sl(H(A, J)_0)$, therefore

$$\operatorname{Lie}(T, \alpha) = sl(T).$$

We now turn to the computations of the tensors. We obtain that $T^\alpha(x, y) = 2\nu q[x, y]$, while

$$\begin{aligned} R^\alpha(x, y) &= [\alpha(x, -), \alpha(y, -)] - [x, y, -] \\ &= [\eta R^{\cdot}_x + \nu q \operatorname{ad} x, \eta R^{\cdot}_y + \nu q \operatorname{ad} y] - \mu \operatorname{ad}[x, y] \\ &= \eta^2 [R^{\cdot}_x, R^{\cdot}_y] + \eta\nu q([R^{\cdot}_x, ad\, y] + [\operatorname{ad} x, R^{\cdot}_y]) + (\nu^2 q^2 - \mu) \operatorname{ad}[x, y]. \end{aligned}$$

If we denote by l_x and r_x the right and left multiplications in the associative algebra A, it is easily checked using (*) that

$$\begin{aligned} [R^{\cdot}_x, R^{\cdot}_y] &= \frac{1}{4}[l_x + r_x, l_y + r_y] + \frac{1}{n}(t(x, -)y - t(y, -)x) \\ &= \frac{1}{4} ad[x, y] + \frac{1}{n}(t(x, -)y - t(y, -)x) \end{aligned}$$

and, since $\operatorname{ad} qx \in \operatorname{Der}(H(A, j)_0, \cdot)$ then $[\operatorname{ad} qx, R^{\cdot}_y] = R^{\cdot}_{q[x,y]}$.

Combining these equalities we get

$$R^\alpha(x, y) = \left(\frac{\eta^2}{4} + \nu^2 q^2 - \mu\right) \operatorname{ad}[x, y] + 2\eta\nu R^{\cdot}_{q[x,y]} + \frac{\eta^2}{n}(t(x, -)y - t(y, -)x).$$

We note that

$$\begin{aligned} &R^\alpha(x, y) = \\ &= [\alpha(x, -), \alpha(y, -)] - \mu \operatorname{ad}[x, y] \in \operatorname{Lie}(T, \alpha) + \operatorname{Der}(T, \alpha) \subseteq \operatorname{Lie}(T, \alpha) \end{aligned}$$

and $\alpha(x,-) \in \mathrm{Lie}(T,\alpha)$, so $\mathrm{Hol}^\alpha(T) \subseteq \mathrm{Lie}(T,\alpha)$. Besides $\mathrm{Hol}^\alpha(T)$ is closed under commutation by $\alpha(x,-)$, but these operators generate $\mathrm{Lie}(T,\alpha)$, which forces $\mathrm{Hol}^\alpha(T)$ to be an ideal of $\mathrm{Lie}(T,\alpha)$. As we saw that $\mathrm{Lie}(T,\alpha) = sl(T)$, it is a simple algebra and as in the Jordan type we may conclude that

$$\begin{cases} T^\alpha(x,y) = 2\nu q[x,y] \\ R^\alpha(x,y) = \left(\frac{\eta^2}{4} + \nu^2 q^2 - \mu\right) \mathrm{ad}[x,y] + 2\eta\nu R_{q[x,y]} \\ \qquad\qquad + \frac{\eta^2}{n}\left(t(x,-)y - t(y,-)x\right) \\ \mathrm{Hol}^\alpha(T) = sl(T). \end{cases}$$

Notice that in case $K = F[q] = F \oplus F$, then $A = B \oplus B^{op}$, with B a central simple associative algebra, and j is the exchange involution. Thus $H(A,j)$ may be identified with B^+ and $S(A,j)$ with the Lie algebra B^- by means of the projection onto the first component. The formulae above remain valid for $T = B_0$ in this case, with $q = 1$ and with t the generic trace of B.

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