nonassociative algebra and its applications

the fourth international conference

edited by

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Octonions and affine connections on spheres

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ABSTRACT: The interplay of the theory of octonions with affine connections on the spheres S^6, S^7 and S^{15} will be presented. Specifically, when these spheres are realized as the reductive homogeneous spaces $S^6 = G_2/SU(3)$, $S^7 = \mathrm{Spin}(7)/G_2$ and $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$, the structure of octonions can be effectively used to determine all left invariant affine connections on them. These connections on S^6 and S^7 are respectively obtained from a two parameters family of the compact vector color algebra and a one parameter family of the compact simple non-Lie Malcev algebra. If $\mathbb{O} = \mathbb{R} \oplus \mathbb{O}_0$ is the algebra of octonions (or Cayley algebra), then all Spin(9)-invariant affine connections on S^{15} are given by a three parameters family of $\mathbb{Z}_2$-graded products on $\mathbb{O}_0 \times \mathbb{O}$, with even part $\mathbb{O}_0$ and odd part $\mathbb{O}$.

1. INTRODUCTION.

Let G be a Lie group acting smoothly and transitively on a manifold M. Given any $p \in M$ and the isotropy subgroup at p: $H = \{g \in G : g \cdot p = p\}$, the homogeneous space $M \simeq G/H$ is said to be *reductive* in case there is a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{1}$$

(where $\mathfrak{h}$ is the Lie subalgebra of $\mathfrak{g}$ which corresponds to the closed subgroup H) and $(\mathrm{Ad}\, H)(\mathfrak{m}) \subseteq \mathfrak{m}$; so that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, and the converse is true if H is connected.

* Supported by the Spanish DGICYT (Pb 94-1311-C03-03) and by the Universidad de La Rioja (API-98/B15).

† Supported by KIAS Grant M98004.

Then the pairs (G, H) and $(\mathfrak{g}, \mathfrak{h})$ are called *reductive pairs*. Notice that for compact H, such a complement $\mathfrak{m}$ always exist (just take any complement of $\mathfrak{h}$ as a vector subspace of $\mathfrak{g}$ and "average" the corresponding projection operator).

An affine connection ∇ on a homogeneous space $M \simeq G/H$ is said to be *invariant* in case the left translation by any element of G is an affine transformation of ∇.

Nomizu ([1, Theorem 8.1]) proved in 1954 the following fundamental theorem for affine connections:

THEOREM 1. *Let $M \simeq G/H$ be a reductive homogeneous space with decomposition (1). Then the set of G-invariant affine connections on M is in bijection with the set of bilinear maps*

$$\alpha : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$$

with $(\mathrm{Ad}\, h)(\alpha(x,y) = \alpha\big((\mathrm{Ad}\, h)(x), (\mathrm{Ad}\, h)(y)\big)$ for any $h \in H$ and any $x, y \in \mathfrak{m}$.

In other words, each invariant affine connection on M is determined by a bilinear product on $\mathfrak{m}$ which admits $\mathrm{Ad}\, H|_{\mathfrak{m}}$ as a subgroup of its group of automorphisms. This implies that $\mathrm{ad}\, \mathfrak{h}|_{\mathfrak{m}}$ is contained in the Lie algebra of derivations, and the converse is valid if H is connected.

Looking at $\mathfrak{m}$ as a module for the group H or the Lie algebra $\mathfrak{h}$ via the adjoint actions, Nomizu's Theorem asserts that the set of invariant affine connections on M is in bijection with the vector space $\mathrm{Hom}_H(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$, or with $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$ if H is connected.

Nomizu proved this Theorem by identifying the tangent space T_pM of M at p with $\mathfrak{m}$ and by defining for each vector field $X \in \mathfrak{m} \subseteq \mathfrak{g}$ a vector field $\tilde{X}$ on a neighborhood of p in M, and then considered for each invariant affine connection ∇ on M the product given by

$$\alpha(X, Y) = \left(\nabla_{\tilde{X}} \tilde{Y}\right)_p .$$

Dealing in a more global way (see [2, Chapter IV]), one can consider for each left-invariant vector field $X \in \mathfrak{g}$ the global vector field X^+ defined on M by $X^+_m = \frac{d}{dt}(\exp tX) \cdot m$. Now, for any invariant affine connection ∇ on M we may identify $\mathfrak{m}$ with the tangent space T_pM by means of $X \mapsto X^+_p$ and associate with ∇ the product defined by

$$\alpha(X, Y) = \left(\nabla_{X^+} Y^+ - [X^+, Y^+]\right)_p . \qquad (2)$$

This too gives a bijection as in Theorem 1.

The two basic tensors associated to any connection ∇, torsion and curvature, are then determined by their values at p (by invariance) and after the identification above of T_pM with $\mathfrak{m}$, they are given by the formulae:

$$\begin{aligned} T(X,Y) &= \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}} \\ R(X,Y)Z &= \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) \\ &\quad - \alpha([X,Y]_{\mathfrak{m}},Z) - [[X,Y]_{\mathfrak{h}},Z] \end{aligned} \tag{3}$$

where for any $X,Y \in \mathfrak{m}$, $[X,Y]_{\mathfrak{m}}$ and $[X,Y]_{\mathfrak{h}}$ denote the projections of the Lie bracket $[X,Y]$ onto $\mathfrak{m}$ and $\mathfrak{h}$, respectively, in the decomposition (1) and α is given by (2).

Two important invariant connections appear immediately associated to any reductive homogeneous space, the *canonical connection*, which corresponds to $\alpha(X,Y) = 0\ \forall X,Y \in \mathfrak{m}$, and the *natural connection*, which corresponds to $\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}}\ \forall X,Y \in \mathfrak{m}$. The natural connection is *symmetric*, that is, it has zero torsion.

According to Borel ([3, Theorem III] and [4, Théorème 3]) the only spheres that appear as homogeneous spaces in nonclassical ways are

$$S^6 = G_2/SU(3), \quad S^7 = \mathrm{Spin}(7)/G_2 \quad \text{and} \quad S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)\,. \tag{4}$$

The three of them are intimately related to the algebra of octonions, and this algebra can be used to determine (through Nomizu's Theorem) the invariant affine connections on them. This has been done in [5,6,7,8,9] and the aim of this work is to give a unified survey of these results.

Although much of the algebraic results on the algebras related to the octonions that will appear in what follows can be given in much more generality (for Cayley-Dickson algebras over arbitrary fields), we shall stick for simplicity to the real case. Thus the term algebra will always refer to a finite-dimensional algebra over $\mathbb{R}$.

We finish this introduction with a technical useful result. In all three homogeneous spaces in (4), $\mathfrak{m}$ in the decomposition (1) is unique by the following result (see [5, Proposition 2.3] or [7, Proposition 4]):

LEMMA 2. *Let $\mathfrak{h}$ be a simple subalgebra of a Lie algebra $\mathfrak{g}$ with* $\dim\mathfrak{g} < 2\dim\mathfrak{h}$. *Then there is a unique subspace $\mathfrak{m}$ of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ and $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$; namely, the orthogonal complement of $\mathfrak{h}$ relative to the Killing form of $\mathfrak{g}$.*

Proof: By Cartan's criterion, $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ for the orthogonal complement $\mathfrak{m}$ to $\mathfrak{h}$ relative to the Killing form. Let π be the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ relative to this decomposition. In case $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{n}$ and $[\mathfrak{h},\mathfrak{n}] \subseteq \mathfrak{n}$ for some subspace $\mathfrak{n}$, $\pi(\mathfrak{n})$ is an $\mathfrak{h}$-submodule of $\mathfrak{h}$, which by simplicity and dimension count has to be trivial. Hence $\mathfrak{n} \subseteq \mathfrak{m}$ and $\mathfrak{n} = \mathfrak{m}$ since they both have the same dimension.

2. INVARIANT CONNECTIONS ON $S^6 = G_2/SU(3)$.

The first sphere to be considered is the six dimensional sphere S^6, realized as a quotient of the compact group of type G_2, which is the group of automorphisms of the algebra of octonions, by its subgroup $SU(3)$. To put this explicitly, some facts concerning the octonions will be recalled first.

The algebra of octonions can be defined as $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ with the multiplication defined by

$$(\alpha + u)(\beta + v) = \big(\alpha\beta - \sigma(u,v)\big) + \big(\alpha v + \bar{\beta} u + u * v\big) \tag{5}$$

for any $\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathbb{C}^3$, where $\sigma : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ is the usual hermitian form given by $\sigma((x_1,x_2,x_3),(y_1,y_2,y_3)) = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$, and the real product $*$ on $\mathbb{C}^3$ is defined by means of

$$\sigma(u, v * w) = \det(u, v, w)$$

for any $u, v, w \in \mathbb{C}^3$. The real algebra $(\mathbb{C}^3, *)$ is related to the so-called "color algebra" introduced in [10] (see [11]).

Given any $x = \alpha + u \in \mathbb{O}$, let $\bar{x} = \bar{\alpha} - u$. Then $t(x) = x + \bar{x}$ and $n(x) = x\bar{x}$ are real numbers and x satisfies the quadratic equation

$$x^2 - t(x)x + n(x)1 = 0\,. \tag{6}$$

Identifying $\mathbb{O}$ with $\mathbb{R}^8$, $n(x)$ turns out to be the euclidean norm in $\mathbb{R}^8$, hence $\mathbb{O}_0 = \{x \in \mathbb{O} : t(x) = 0\} = \mathbb{R}i \oplus \mathbb{C}^3$ can be identified with the euclidean space $\mathbb{R}^7$ and the six dimensional sphere S^6 lives inside $\mathbb{O}_0$ as $S^6 = \{x \in \mathbb{O}_0 : n(x) = 1\}$.

The group of automorphisms $\operatorname{Aut}\mathbb{O}$ is the compact Lie group of type G_2. Because of (6) it preserves the standard involution $x \mapsto \bar{x}$, the trace t and the norm n, so that it preserves $\mathbb{O}_0$ and $S^6 \subseteq \mathbb{O}_0$. Moreover, the action on S^6 is transitive (see [5, Theorem 6.5] or [12, exercise 6.9]).

Take the element $p = i \in S^6$. Any automorphism fixing i fixes $\mathbb{C}$ elementwise, so that it preserves $\mathbb{C}^3$, which is the orthogonal complement of $\mathbb{C}$ relative to n. By (5), it is an automorphism of the algebra $(\mathbb{C}^3, *)$, it is $\mathbb{C}$-linear on $\mathbb{C}^3$ and belongs to the unitary group. The definition of the product $*$ forces its determinant (over $\mathbb{C}$) to be 1. Hence it belongs to $SU(3)$, and conversely. Thus, the isotropy subgroup of the induced action $G_2 \times S^6 \to S^6$ at i is $SU(3)$ and $S^6 \simeq G_2/SU(3)$.

The Lie algebra of $G_2 = \operatorname{Aut}\mathbb{O}$ is $\mathfrak{g}_2 = \operatorname{Der}\mathbb{O}$, and the Lie algebra of the closed subgroup $SU(3)$ is $su(3) = \{d \in \operatorname{Der}\mathbb{O} : d(\mathbb{C}) = 0\}$. By Lemma 2, there is a unique complement of $su(3)$ invariant under the adjoint action of $su(3)$.

Now, by [13, III.8] we know that $\operatorname{Der}\mathbb{O}$ is the linear span of the maps

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] = L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y]\,,$$

where L_x and R_x denote the left and right multiplications by x. For $x, y \in \mathbb{O}_0$, the restriction $D_{x,y}|_{\mathbb{O}_0}$ is twice the map $d(x,y)$ considered in [5, (2.11)] and [6] (see [8, Lemma 6]).

For any $u \in \mathbb{C}^3$ $D_{i,u}(i) = i(ui) - ui^2 + i(iu) - i^2u + (iu)i - i^2u = 4u$. From this it follows that

$$\mathfrak{g}_2 = su(3) \oplus D_{i,\mathbb{C}^3},$$

and for any $D \in su(3)$ and $u \in \mathbb{C}^3$, $[D, D_{i,u}] = D_{D(i),u} + D_{i,D(u)} = D_{i,D(u)} \in D_{i,\mathbb{C}^3}$, so that $\mathfrak{m} = D_{i,\mathbb{C}^3} = su(3)^\perp$.

In $\mathbb{C}^3$ consider the new product given by $u \circ v = i(u * v)$. Actually, the map $(\mathbb{C}^3, \circ) \to (\mathbb{C}^3, *)$, $u \mapsto iu$ is an isomorphism of algebras. Now the identification of $\mathfrak{m}$ with the tangent space of S^6 at i, which is the orthogonal complement of $\mathbb{R}i$ in $\mathbb{O}_0$, that is $\mathbb{C}^3$, is given by

$$X = D_{i,u} \mapsto X_p^+ = \frac{d}{dt}|_{t=0}(\exp tD_{i,u})(i) = D_{i,u}(i) = 4u.$$

This is not just a vector space isomorphism (see [5, Proposition 4.2]):

Proposition 3. *The linear map $(\mathfrak{m}, [\,,\,]_\mathfrak{m}) \to (\mathbb{C}^3, \circ)$, given by $D_{i,u} \mapsto 4u$, is an isomorphism of algebras and of $su(3)$-modules.*

Therefore, using that $SU(3)$ is connected and this Proposition we get

$$\mathrm{Hom}_{SU(3)}(\mathfrak{m} \otimes_\mathbb{R} \mathfrak{m}, \mathfrak{m}) \cong \mathrm{Hom}_{su(3)}(\mathbb{C}^3 \otimes_\mathbb{R} \mathbb{C}^3, \mathbb{C}^3)$$

and this is a real vector space of dimension 2, something that is checked by extending scalars up to $\mathbb{C}$ ([5, Section 5]). Since for any $\mu \in \mathbb{C}$, the map $\mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$, $(u,v) \mapsto \mu(u \circ v)$ is $su(3)$-invariant, it follows that $\mathrm{Hom}_{su(3)}(\mathbb{C}^3 \otimes_\mathbb{R} \mathbb{C}^3, \mathbb{C}^3)$ is spanned over $\mathbb{C}$ by the product $\circ$. That is, it is spanned over $\mathbb{R}$ by the products $(u,v) \mapsto u \circ v$ and $(u,v) \mapsto i(u \circ v)$. It should be noticed here that the multiplication by i on $\mathbb{C}^3 \simeq T_iS^6$ corresponds to the classical invariant almost complex structure on S^6.

We have arrived at:

Theorem 4. *There is a two parameters family of G_2-invariant affine connections on S^6.*

Through the isomorphism in Proposition 3, the formulae (3) for the torsion and curvature of the connection determined by the product $(u,v) \mapsto \mu(u \circ v)$ $(\mu \in \mathbb{C})$ are given by ([6, Theorem 9]):

$$\begin{aligned} T(u,v) &= (2\mu - 1)(u \circ v), \\ R(u,v)w &= \left(|\mu|^2 - \frac{1}{4}\right)(u \circ (v \circ w) - v \circ (u \circ w)) \\ &\qquad - \left(\mu - \frac{1}{2}\right)(u \circ v) \circ w + n(v,w)u - n(u,w)v \end{aligned}$$

where $n(u,v) = \frac{1}{2}(n(u+v) - n(u) - n(v))$ (the real part of $\sigma(u,v)$).

The only symmetric G_2-invariant affine connection is the natural one ($\mu = \frac{1}{2}$), and for it the curvature has a very simple expression: $R(u,v)w = n(v,w)u - n(u,w)v$ ([6, Corollary 10]). This natural connection is the usual Riemannian connection on S^6.

3. INVARIANT CONNECTIONS ON $S^7 = \mathrm{Spin}(7)/G_2$.

There is a triple product defined on the octonions with some nice properties, which is given by

$$\{x,y,z\} = (x\bar{y})z \tag{7}$$

for any $x,y,z \in \mathbb{O}$ (see [14] for the main features of this product). The group of automorphisms of this triple product turns out to be isomorphic to the compact group $B_3 = \mathrm{Spin}(7)$ under its spin representation. The action of this automorphism group is transitive on the seven dimensional sphere $S^7 = \{x \in \mathbb{O} : n(x) = 1\}$. There is a distinguished element in S^7, the unit element $1 \in \mathbb{O}$, and the isotropy subgroup at 1 is the subgroup of those automorphisms of the triple product that fix 1. From (7) with $y = 1$ it follows that this isotropy subgroup is exactly $G_2 = \mathrm{Aut}\,\mathbb{O}$.

The Lie algebra $\mathfrak{b}_3$ of B_3 is the Lie algebra of derivations of the triple product in (7), and the Lie subalgebra associated to the isotropy subgroup at 1 is the subalgebra of the derivations of $\mathbb{O}$ (these are the derivations of the triple product that annihilate 1).

¿From the alternative laws we have for any $u,x,y \in \mathbb{O}$:

$$\begin{aligned} L_u(xy) &= (L_u + R_u)(x)y - xL_u(y), \\ R_u(xy) &= -R_u(x)y + x(L_u + R_u)(y). \end{aligned} \tag{8}$$

Hence

$$\begin{aligned} (L_u + 2R_u)(xy) &= \mathrm{ad}_u(x)y + x(L_u + 2R_u)(y), \\ \mathrm{ad}_u(xy) &= (L_u + 2R_u)(x)y - x(2L_u + R_u)(y) \end{aligned} \tag{9}$$

where $\mathrm{ad}_u(x) = [u,x] = ux - xu$. For any $u \in \mathbb{O}_0$, let $E_u = L_u + 2R_u$. Also notice that $(2L_u + R_u)(\bar{x}) = -\overline{E_u(x)}$ for any $u \in \mathbb{O}_0$ and any $x \in \mathbb{O}$, since $\bar{u} = -u$. Hence, (9) implies that

$$\begin{aligned} E_u(\{x,y,z\} = E_u((x\bar{y})z) &= \mathrm{ad}_u(x\bar{y})z + (x\bar{y})E_u(z) \\ &= (E_u(x)\bar{y})\,z - (x(2L_u + R_u)(\bar{y})) + (x\bar{y})E_u(z) \\ &= \{E_u(x),y,z\} + \{x,E_u(y),z\} + \{x,y,E_u(z)\} \end{aligned}$$

for any $u \in \mathbb{O}_0$ and any $x,y,z \in \mathbb{O}$. Thus, $E_u \in \mathfrak{b}_3$ for any $u \in \mathbb{O}_0$. Moreover, since $\{x,x,y\} = n(x)y$ for any $x,y \in \mathbb{O}$, it follows that $\mathfrak{b}_3 \subseteq o(\mathbb{O},n)$, the orthogonal Lie algebra relative to the euclidean norm n; hence for any $D \in \mathfrak{b}_3$, $D(1) = u \in \mathbb{O}$ and

$D = (D - \frac{1}{3}E_u) + \frac{1}{3}E_u$, with $(D - \frac{1}{3}E_u)(1) = 0$, so that $D - \frac{1}{3}E_u \in \operatorname{Der}\mathbb{O} = \mathfrak{g}_2$. Therefore

$$\mathfrak{b}_3 = \mathfrak{g}_2 \oplus E_{\mathbb{O}_0}$$

and for any $D \in \mathfrak{g}_2$ and $u \in \mathbb{O}_0$, $[D, E_u] = [D, L_u + 2R_u] = L_{D(u)} + 2R_{D(u)} = E_{D(u)}$, so that $\mathfrak{m} = E_{\mathbb{O}_0}$ is the orthogonal complement to $\mathfrak{g}_2$ relative to the Killing form of $\mathfrak{b}_3$, because of Lemma 2.

Now, the identification of $\mathfrak{m} = E_{\mathbb{O}_0}$ with the tangent space to S^7 at 1: $T_1S^7 = \mathbb{O}_0$, is given by

$$X = E_u \mapsto X_1^+ = \frac{d}{dt}|_{t=0}(\exp tE_u)(1) = E_u(1) = 3u\,.$$

For any $u, v \in \mathbb{O}_0$, [7, Lemma 2] shows that

$$\begin{aligned}[E_u, E_v] &= [L_u + 2R_u, L_v + 2R_v] \\ &= 2D_{u,v} - (L_{[u,v]} + 2R_{[u,v]}) = 2D_{u,v} - E_{[u,v]}\,.\end{aligned}$$

This immediately gives the following:

Proposition 5. *The linear map $(\mathfrak{m}, [\,,\,]_{\mathfrak{m}}) \to (\mathbb{O}_0, -\frac{1}{3}[\,,\,])$, given by $E_u \mapsto 3u$, is an isomorphism of algebras and of $\mathfrak{g}_2$-modules.*

Notice that the algebra $(\mathbb{O}_0, -\frac{1}{3}[\,,\,])$ is isomorphic to $(\mathbb{O}_0, [\,,\,])$, which is a central simple non-Lie Malcev algebra. For any $x, y \in \mathbb{O}_0$, let us write $x \circ y = -\frac{1}{3}[x, y]$.

Using Proposition 5 and [7, Theorem 8] we get

$$\operatorname{Hom}_{G_2}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m}) \cong \operatorname{Hom}_{\mathfrak{g}_2}(\mathbb{O}_0 \otimes_{\mathbb{R}} \mathbb{O}_0, \mathbb{O}_0)$$

and this last real vector space is spanned by the map given by $x \otimes y \mapsto [x, y]$ (or by $x \otimes y \mapsto x \circ y$). Then:

Theorem 6. *There is a one parameter family of* Spin(7)*-invariant affine connections on S^7.*

Through the isomorphism in Proposition 5, the formulae for the torsion and curvature of the connection determined by the product given by $(u, v) \mapsto \mu(u \circ v)$ $(\mu \in \mathbb{R})$ are given, with the same computations as in [8, Theorem 7] but taking into account that $[u, v] = 2(u \times v)$ $\forall u, v \in \mathbb{O}_0$, by

$$\begin{aligned} T(u, v) &= (2\mu - 1)u \circ v, \\ R(u, v)w &= (1 - \mu - 2\mu^2)(u \circ v) \circ w + \frac{4}{3}(\mu^2 - 1)\big(n(u, w)v - n(v, w)u\big)\,. \end{aligned}$$

As for S^6, the only symmetric Spin(7)-invariant affine connection is the natural one $(\mu = \frac{1}{2})$ and for it the curvature tensor has the same very simple expression, since this is again the usual Riemannian connection on S^7.

4. INVARIANT CONNECTIONS ON $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$.

Let us consider now the nine dimensional euclidean space $\mathbb{R}^9 = \mathbb{R} \times \mathbb{O}$. Identify $\mathbb{O}$ with the subspace $0 \times \mathbb{O}$ and let $e = (1,0)$. Then $\mathbb{R}^9$ is the orthogonal direct sum of $\mathbb{R}e$ and $\mathbb{O}$. The linear map

$$\rho : \mathbb{R} \times \mathbb{O} \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{O} \times \mathbb{O}),$$
$$\alpha e + x \mapsto \begin{pmatrix} \alpha & L_x \\ L_{\bar{x}} & -\alpha \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $x \in \mathbb{O}$ (that is $\rho(\alpha e + x)(u, v) = (\alpha u + xv, \bar{x}u - \alpha v)$), induces an isomorphism, also denoted by ρ, of the even Clifford algebra of $\mathbb{R}^9$ onto $\mathrm{End}_{\mathbb{R}}(\mathbb{O}\times\mathbb{O})$. The corresponding spin group Spin(9) will be identified in what follows with its image under ρ. It is not difficult to see that Spin(9) acts transitively on the sphere $S^{15} = \{(x, y) \in \mathbb{O} \times \mathbb{O} : n(x) + n(y) = 1\}$ (see [9] or [12, Theorem 14.79], although the way it is done in this last reference is slightly different).

By the well known triality principle (see [12, 14.19]) this allows to identify Spin(8) with the subgroup of Spin(9) consisting of the elements

$$\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} \in \mathrm{End}_{\mathbb{R}}(\mathbb{O} \times \mathbb{O})$$

where φ_1, φ_2 are orthogonal transformations of $\mathbb{O}$ and there is a third orthogonal transformation φ_0 of $\mathbb{O}$ satisfying $\varphi_1(xy) = \varphi_0(x)\varphi_2(y)$ for any $x, y \in \mathbb{O}$.

Given an automorphism φ of the triple product in (7), we have for any $x, y \in \mathbb{O}$:

$$\varphi(xy) = \varphi(\{x, 1, y\}) = \{\varphi(x), \varphi(1), \varphi(y)\} = \tilde{\varphi}(x)\varphi(y)\,,$$

where $\tilde{\varphi}(x) = \varphi(x)\overline{\varphi(1)}$. Then $\varphi(1) = \tilde{\varphi}(1)\varphi(1)$, so that $\tilde{\varphi}(1) = 1$. Moreover, all this implies for any $x, y, z \in \mathbb{O}$ that

$$(\varphi(x)\overline{\varphi(y)})\varphi(z) = \varphi(\{x, y, z\}) = \varphi((x\bar{y})z) = \tilde{\varphi}(x\bar{y})\varphi(z)\,,$$

so that $\tilde{\varphi}(xy) = \varphi(x)\overline{\varphi(\bar{y})}$ for any $x, y \in \mathbb{O}$. Let $\hat{\varphi}$ be defined by $\hat{\varphi}(x) = \overline{\varphi(\bar{x})}$. Then $\tilde{\varphi}(xy) = \varphi(x)\hat{\varphi}(y)$ for any $x, y \in \mathbb{O}$ and the group Spin(7), which is the group of automorphisms of the triple product, embeds in Spin(8) through

$$\varphi \mapsto \begin{pmatrix} \tilde{\varphi} & 0 \\ 0 & \hat{\varphi} \end{pmatrix}.$$

Let H be the isotropy subgroup of the action of Spin(9) on S^{15} at the point $(1,0) \in \mathbb{O} \times \mathbb{O}$. Then the arguments above show that, since $\tilde{\varphi}(1) = 1$, H contains a subgroup K isomorphic to Spin(7). Thus $\mathrm{Spin}(9)/K \to S^{15} : gK \mapsto g \cdot (1,0)$ is a covering map, and since S^{15} is simply connected it follows by dimension count that this is a bijection, so that $H = K \cong \mathrm{Spin}(7)$ (see [12, Theorem 14.79], or [9] for a proof in a wider context).

The Lie algebra $\mathfrak{b}_4$ of Spin(9) is graded over $\mathbb{Z}_2$ in a natural way:

$$\mathfrak{b}_4 = \mathfrak{g}_0 \oplus \mathfrak{g}_1 ,$$

where $\mathfrak{g}_0$ is the Lie algebra of Spin(8), spanned by the elements

$$\rho(x)\rho(y) - \rho(y)\rho(x) = \begin{pmatrix} L_x L_{\bar y} - L_y L_{\bar x} & 0 \\ 0 & L_{\bar x} L_y - L_{\bar y} L_x \end{pmatrix} \in \mathrm{End}_{\mathbb{R}}(\mathbb{O} \times \mathbb{O})$$

for $x, y \in \mathbb{O}$, and where $\mathfrak{g}_1$ is spanned by the elements

$$P_x = \rho(e)\rho(x) = \begin{pmatrix} 0 & L_x \\ -L_{\bar x} & 0 \end{pmatrix}$$

for $x \in \mathbb{O}$.

Notice that

$$\mathrm{span}\,\langle L_x L_{\bar y} - L_y L_{\bar x} : x, y \in \mathbb{O}\rangle = \mathrm{span}\,\langle L_{\bar x} L_y - L_{\bar y} L_x : x, y \in \mathbb{O}\rangle$$

is the orthogonal Lie algebra $o(\mathbb{O}, n)$. Now, by local triality ([13, Theorem 3.31]), for any $d \in o(\mathbb{O}, n)$, there are $d', d'' \in o(\mathbb{O}, n)$ such that for any $x, y \in \mathbb{O}$

$$d(xy) = d'(x)y + xd''(y) . \tag{10}$$

Using (8) it can be checked that $(L_x L_{\bar y} - L_y L_{\bar x})'' = L_{\bar x} L_y - L_{\bar y} L_x$, thus

$$\mathfrak{g}_0 = \left\{ M_d = \begin{pmatrix} d & 0 \\ 0 & d'' \end{pmatrix} : d \in o(\mathbb{O}, n) \right\} .$$

The (1,2)-entry of

$$\left[\begin{pmatrix} d & 0 \\ 0 & d'' \end{pmatrix}, \begin{pmatrix} 0 & L_x \\ L_{-\bar x} & 0 \end{pmatrix} \right]$$

is $dL_x - L_x d''$, which by (10) is exactly $L_{d'(x)}$. Hence

$$[M_d, P_x] = P_{d'(x)} \tag{11}$$

for any $d \in o(\mathbb{O}, n)$ and $x \in \mathbb{O}$.

The Lie subalgebra of the isotropy subgroup of the action of Spin(9) on S^{15} at the point $(1, 0)$ is

$$\mathfrak{h} = \{M_d : d \in o(\mathbb{O}, n) \text{ and } d(1) = 0\} = \{M_d : d \in o(\mathbb{O}_0, n)\} .$$

¿From [13, (3.79)] we have

$$o(\mathbb{O}, n) = \mathrm{Der}\,\mathbb{O} \oplus L_{\mathbb{O}_0} \oplus R_{\mathbb{O}_0} = \mathrm{Der}\,\mathbb{O} \oplus \mathrm{ad}_{\mathbb{O}_0} \oplus T_{\mathbb{O}_0} \tag{12}$$

where $T_x = \frac{1}{2}(L_x + R_x)$ for any $x \in \mathbb{O}$. Since for any $d \in \operatorname{Der}\mathbb{O} \oplus \operatorname{ad}_{\mathbb{O}_0}$, $d(1) = 0$, it follows from (12) that

$$o(\mathbb{O}_0, n) = \operatorname{Der}\mathbb{O} \oplus \operatorname{ad}_{\mathbb{O}_0} .$$

Now from the flexible law we get $[\operatorname{ad}_x, T_y] = T_{\operatorname{ad}_x(y)} = T_{[x,y]}$ for any $x, y \in \mathbb{O}$, so that

$$[d, T_x] = T_{d(x)} \tag{13}$$

for any $d \in o(\mathbb{O}_0, n)$ and $x \in \mathbb{O}_0$, and hence $T_{\mathbb{O}_0}$ is a subspace of $o(\mathbb{O}, n)$ invariant under the adjoint action of $o(\mathbb{O}_0, n) = \operatorname{Der}\mathbb{O} \oplus \operatorname{ad}_{\mathbb{O}_0}$. If we write

$$N_x = \begin{pmatrix} T_x & 0 \\ 0 & T_x'' \end{pmatrix} = \begin{pmatrix} T_x & 0 \\ 0 & \frac{1}{2}R_x \end{pmatrix} \quad \text{(because of (8),)}$$

we arrive at the decomposition

$$\mathfrak{b}_4 = \mathfrak{h} \oplus N_{\mathbb{O}_0} \oplus P_{\mathbb{O}} .$$

Besides, the subspace $\mathfrak{m} = N_{\mathbb{O}_0} \oplus P_{\mathbb{O}}$ is $\mathfrak{h}$-invariant, because of (11) and since $[M_d, N_x] = N_{d(x)}$ by (13) for any $d \in o(\mathbb{O}_0, n)$ and any $x \in \mathbb{O}_0$. From Lemma 2 we conclude that $\mathfrak{m} = N_{\mathbb{O}_0} \oplus P_{\mathbb{O}}$ is the orthogonal complement of $\mathfrak{h}$ relative to the Killing form of $\mathfrak{b}_4$.

The identification of $\mathfrak{m} = N_{\mathbb{O}_0} \oplus P_{\mathbb{O}}$ with the tangent space to S^{15} at $(1,0)$, $T_{(1,0)}S^{15} = \mathbb{O}_0 \times \mathbb{O}$ is given by

$$X = N_x + P_y \mapsto X^+_{(1,0)} = (N_x + P_y)(1,0) = \begin{pmatrix} T_x & L_y \\ -L_{\bar{y}} & \frac{1}{2}R_x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ -\bar{y} \end{pmatrix} .$$

As for the multiplication among the different elements involved, notice that

$$\begin{aligned} &[N_x, P_y] = P_{(T_x)'(y)} = \frac{1}{2}P_{xy} \quad \text{because of (11) and (8)}, \\ &[N_{x_1}, N_{x_2}] \in \mathfrak{h} \quad \text{since } [T_{x_1}, T_{x_2}](1) = 0, \\ &[P_{y_1}, P_{y_2}]_{\mathfrak{m}} = -N_{y_1\bar{y}_2 - y_2\bar{y}_1} \quad \text{(see [9])}, \end{aligned}$$

for any $x, x_1, x_2 \in \mathbb{O}_0$ and $y, y_1, y_2 \in \mathbb{O}$. From this we get immediately:

PROPOSITION 7. *The linear map*

$$\begin{aligned} (\mathfrak{m}, [\,,\,]_{\mathfrak{m}}) &\to (\mathbb{O}_0 \times \mathbb{O}, \diamond) : \\ N_x + P_y &\mapsto (x, -\bar{y}) , \end{aligned}$$

where

$$(x_1, y_1) \diamond (x_2, y_2) = (\bar{y}_2 y_1 - \bar{y}_1 y_2, \frac{1}{2}(y_1 x_2 - y_2 x_1)$$

for any $x_1, x_2 \in \mathbb{O}_0$, $y_1, y_2 \in \mathbb{O}$, *is an isomorphism of algebras and of* $\mathfrak{h} = o(\mathbb{O}_0, n)$*-modules, where* $\mathbb{O}_0$ *is given the natural structure of* $o(\mathbb{O}_0, n)$*-module, while* $\mathbb{O}$ *is the irreducible* $o(\mathbb{O}_0, n)$*-module given by the spin representation:* $d \cdot x = d''(x)$ *for any* $d \in o(\mathbb{O}_0, n)$ and $x \in \mathbb{O}$.

Using Proposition 7 one gets

$$\mathrm{Hom}_H(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m}) \cong \mathrm{Hom}_{\mathfrak{h}}\left((\mathbb{O}_0 \times \mathbb{O}) \otimes_{\mathbb{R}} (\mathbb{O}_0 \times \mathbb{O}), \mathbb{O}_0 \times \mathbb{O}\right),$$

and this last space is determined in [9]. It has dimension 3 and it is spanned by the products:

$$\begin{aligned}
(x_1, y_1) \diamond_0 (x_2, y_2) &= (\bar{y}_2 y_1 - \bar{y}_1 y_2, 0), \\
(x_1, y_1) \diamond_1 (x_2, y_2) &= (0, y_1 x_2), \\
(x_1, y_1) \diamond_2 (x_2, y_2) &= (0, y_2 x_1).
\end{aligned}$$

THEOREM 8. *There is a three parameters family of* Spin(9)*-invariant affine connections on* S^{15}.

Through the isomorphism given in Proposition 7 it is straightforward, although tedious, to compute the torsion and curvature tensors, as we did for S^6 and S^7, of the invariant affine connection determined by the product

$$\begin{aligned}
&(x_1, y_1) \star (x_2, y_2) \\
&\quad = \gamma_0 (x_1, y_1) \diamond_0 (x_2, y_2) + \gamma_1 (x_1, y_1) \diamond_1 (x_2, y_2) - \gamma_2 (x_1, y_1) \diamond_2 (x_2, y_2) \qquad (14) \\
&\quad = (\gamma_0(\bar{y}_2 y_1 - \bar{y}_1 y_2), \gamma_1 y_1 x_2 - \gamma_2 y_2 x_1).
\end{aligned}$$

We will do it only for the torsion tensor:

$$\begin{aligned}
&T\left((x_1, y_1), (x_2, y_2)\right) \\
&\quad = (x_1, y_1) \star (x_2, y_2) - (x_2, y_2) \star (x_1, y_1) - (x_1, y_1) \diamond (x_2, y_2) \\
&\quad = \left((2\gamma_0 - 1)(\bar{y}_2 y_1 - \bar{y}_1 y_2), (\gamma_1 + \gamma_2 - \frac{1}{2})(y_1 x_2 - y_2 x_1) \right).
\end{aligned}$$

Hence there is a whole one parameter family of symmetric invariant affine connections: those determined by $\gamma_0 = \frac{1}{2} = \gamma_1 + \gamma_2$. The natural connection corresponds to $\gamma_0 = \frac{1}{2}$, $\gamma_1 = \gamma_2 = \frac{1}{4}$.

The usual Riemannian metric on S^{15} (which is invariant under the whole orthogonal group $O(16)$, so in particular it is Spin(9)-invariant) is given in $T_{(1,0)}S^{15} = \mathbb{O}_0 \times \mathbb{O}$ by $\langle (x_1, y_1), (x_2, y_2) \rangle = n(x_1, x_2) + n(y_1, y_2)$ (the usual scalar product on $\mathbb{R}^{15} = \mathbb{O}_0 \times \mathbb{O}$). According to [1, (13.1)] (where a minus sign is missing!) the corresponding Riemannian connection is determined by the product (through Proposition 7):

$$\frac{1}{2}(x_1, y_1) \diamond (x_2, y_2) + U\left((x_1, y_1), (x_2, y_2)\right)$$

where U is determined by

$$\begin{aligned}2\langle U\left((x_1,y_1),(x_2,y_2)\right),(x_3,y_3)\rangle\\ = \langle (x_1,y_1),(x_3,y_3)\diamond(x_2,y_2)\rangle + \langle (x_2,y_2),(x_3,y_3)\diamond(x_1,y_1)\rangle .\end{aligned}$$

Using that $n(ab,c)=n(b,\bar{a}c)=n(a,c\bar{b})$ and that $n(a,b)=n(\bar{a},\bar{b})$ for any $a,b,c\in\mathbb{O}$, it is easy to check that

$$U\left((x_1,y_1),(x_2,y_2)\right)=\left(0,\frac{3}{4}(y_1x_2+y_2x_1)\right)$$

whence the usual Riemannian connection on S^{15} corresponds to the product (14) with $\gamma_0=\frac{1}{2}$, $\gamma_1=1$ and $\gamma_2=\frac{1}{2}$ (see [9] for details).

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