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AGGREGATION OF PREFERENCES IN CRISP AND FUZZY SETTINGS: FUNCTIONAL EQUATIONS LEADING TO POSSIBILITY RESULTS

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We analyze various models introduced in social choice to aggregate individual preferences. We show that on the basis of most of these models there is a system of functional equations such that, in many cases, the origin of impossibility results in a social choice model is the non-existence of a solution for the corresponding system. Among the functional equations considered, we pay a particular attention to general means and associativity, proving that the existence of an associative bivariate mean is equivalent to the existence of a semilatticial partial order. This key result allows us to explain how the knowledge of associative bivariate means can be used to solve social choice paradoxes. In our analysis we deal both with crisp and fuzzy settings.

Keywords: Individual and social preferences; aggregation of preferences; social choice models; functional equations; fuzzy preferences.

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1. Introduction

The present paper intends to be a further contribution to the extensive literature on aggregation of individual *preferences*.

Individuals in a society are likely to have diverse views and preferences on a particular matter or issue; how to aggregate them into a reasonable or acceptable social preference or choice has been the concern of *Social choice theory*. Traditionally, election (voting, counting and decision based on majority of votes) has been one of the means by which a democratic society articulated its collective preference. However, when the individual preferences are expressed as comparisons between the available alternatives (ordinal preferences), the aggregation process becomes more complex. In fact, even apparently reasonable criteria that one tries to assign to a social choice (aggregation) rule prove to be inconsistent collectively.

We will analyze the so-called *social choice paradox*.

Social choice paradoxes appear when a society is unable to find a rule to aggregate the individual preferences into a *collective social preference* accomplishing a mild list of restrictions. An evident example of a common sense restriction is the *respect of unanimity* so that if every individual of a society declares that the option A is better than the option B, then the society as a whole should also declare that A is better than B.

This general setting was launched in the pioneer work by Arrow.[1]

Arrow formulated it as an *impossibility theorem* and proved it using a *combinatorial approach*, showing that under a small set of conditions (known as the *Arrovian model*), and for a finite society with at least two individuals, that has to rank a set of at least three alternatives, there is *no way* to aggregate the individual rankings into a social one accomplishing the previously imposed conditions of the model. The impact of Arrow's impossibility result led other researchers in social choice to look for alternative models to aggregate preferences, in the search for *possibility* results (also known as *resolutions of the social choice paradox*). This obviously implies that the set of conditions to be imposed to the aggregation rules should be modified. Nevertheless, in this direction new (and perhaps discouraging) impossibility results were obtained.[2–4]

A resolution of the social choice paradox was finally achieved and subsequently developed in a *topological approach* introduced by Chichilnisky in a series of papers published mainly during the 1980s.[5–7]

In this new topological context it is *sometimes* (but *not always!*) possible to get rid of paradoxical situations. A social choice rule satisfying certain reasonable conditions that are characteristic of the *Chichilnisky model* and appear as an alternative to those in the Arrovian model *may exist in certain cases.*

As a matter of fact, the *existence* of a social choice rule is strongly *conditioned upon the topological properties* of the spaces involved.

Not only the topological approach introduced by Chichilnisky could lead to possibility results. Indeed, a new *algebraical approach* appeared in the 1990's and, in a somewhat analog way to the topological approach, social aggregation rules may appear here in certain cases, now depending on the *algebraical* structure of the space of preferences.[8]

Although this new approach can be studied independently, it also furnishes further information on the topological one: The main reason is that the usual topological models issued by Chichilnisky have, associated in a natural way, a family of algebraical models based on homotopy groups of the space of preferences, so that these associated algebraical models actually control the topological model, furnishing full information about the existence or not of a social choice rule in the former (topological) model.[8,9]

These three main approaches (combinatorial, topological and algebraical) were all introduced in the *crisp* (i.e. non-fuzzy) *setting* in which the sets of preferences, alternatives and aggregation maps are crisp.

Still in the crisp setting, other different contexts that could also lead to some possibility results appear through *generalizations* of the aforementioned approaches (combinatorial, topological or algebraical). Among them, the extensions of the Arrow or Chichilnisky models to *infinite* populations play an important role.[10–12]

In a sense, this context and the topological or algebraical models share the ideas of *imposing new conditions* to the preferences involved, as well as working on some sets that are *endowed with some extra structure* (e.g.: topological or algebraical).

Somewhat related to this last direction of considering restrictions due to extra structure, a *second main setting* to aggregate preferences has been studied in the literature. This is the *fuzzy* setting, where possibility results may actually appear on various *fuzzy* contexts or aggregation models, that capture the idea of a society where each individual as well as the whole society express its preferences in a *vague, imprecise or uncertain* manner. Thus, for instance, we may suppose that the individual preferences are framed by means of some kind of fuzzy binary relations instead of the total preorders usually arising in the crisp models. Also, we may consider that the social aggregation rules are fuzzy maps. Among many others, there are a large variety of papers where those fuzzy approaches have been considered in the literature.[13–18] Some of these approaches are inspired by some models that are of common use in the crisp setting, as the Arrovian one. Having these (crisp) models in mind, several axiomatic approaches were built in order to understand rational decision malting in a fuzzy environment.[13,15]

In the present paper we shall pay attention to some aspects of recent models introduced in the literature to achieve some possibility qresults. First we shall review the crisp setting, and then we will pay attention to the fuzzy setting.

In the search of social rules coming from models arising in both settings (crisp and fuzzy), some *functional equations* appear in a natural way. As we shall comment later on, the possibility or impossibility results that could appear associated to various social choice models are directly related to the solvability of some system of functional equations.

Intending to furnish a *panoramical review* about the mathematical origin of social paradoxes, mainly due to the lack of solutions of some system of equations that characterizes a social choice model, in the present work we analyze those functional equations, paying a particular attention to general means and the functional equation of associativity, showing that the knowledge of suitable associative bivariate means solves the social choice paradox in a wide variety of models, both in the crisp and fuzzy settings.

The *structure of the paper* goes as follows: In the section of notation and preliminaries (Sec. 2) we shall review the main features of the most important models (combinatorial, topological, and algebraical) arising in the crisp setting. In Sec. 3 we shall study systems of functional equations that are directly related to the models previously introduced in Sec. 2. Among these systems of functional equations, we will pay an special attention to general means (Sec. 4) and associative maps (Sec. 5). In section 6 we shall consider aggregation models introduced in the fuzzy setting. We will also analyze the associated systems of functional equations, and compare the results with the ones obtained in Secs. 3 to 5 for the crisp setting. Further comments about other functional equations arising in social choice models (Sec. 7) and a final section of conclusions (Sec. 8) will close the paper.

2. Notation and Preliminaries

2.1. *Combinatorial models in the crisp setting*

Let Y be a nonempty set of alternatives on which the individuals of a society may define their preferences. In the main *combinatorial* models arising in the crisp setting we suppose that the society is finite, consisting of $N \geq 2$ individuals.[1,3,4,10]

We shall understand here that each individual preference is a *total order* on Y. Let X be the set of all the possible preferences that could be defined on Y (e.g.: if Y is finite, $X = \Pi(Y)$, the set of all the possible permutations of Y).

A *profile* of preferences is an element $(x_1, x_2, \ldots, x_N) \in X^N$ such that each x_i represents an element of X that could be identified with the individual preference of the agent i (where $i \in \{1, \ldots, N\}$). A *social aggregation N-rule* is a map $F_N : X^N \longrightarrow X$.

The most classical combinatorial model is the well known *Arrovian model*, issued in the 1950's, whose restrictions are the following ones:

- (C1) *Whole domain*: The N-rule F_N is defined on the whole set X^N.
- (C2) *Rank restriction*: The N-rule F_N takes values on X. (In other words, for every profile $(x_1, x_2, \ldots, x_N) \in X^N$, $F_N(x_1, x_2, \ldots, x_N) \in X$ is a total order on Y).
- (C3) *Unanimity or Pareto efficiency*: If an alternative $a \in Y$ is ranked above another alternative $b \in Y$ for all the individual preferences of a profile $(x_1, x_2, \ldots, x_N) \in X^N$, then the social preference $F_N(x_1, x_2, \ldots, x_N) \in X$ also ranks a above b. Notice that, in particular, this implies that $F_N(x, x, \ldots, x) = x$ for every $x \in X$).

(C4) *Non-dictatorship*: The N-rule F_N is not a projection over one of its components, that is, there is no $i \in \{1, \dots, N\}$ such that $F_N(x_1, \dots, x_N) = x_i$, for every profile $(x_1, x_2, \dots, x_N) \in X^N$. (This means, roughly speaking, that there is no individual i whose preferences always prevail).

(C5) *Independence of irrelevant alternatives*: For two preference profiles $(x_1, x_2, \dots, x_N)$ and $(y_1, y_2, \dots, y_N) \in X^N$ such that for all individuals $i \in \{1, \dots, N\}$ the alternatives a and $b \in Y$ have the same order in x_i and y_i, it holds that the alternatives a and b also have the same order in $F_N(x_1, x_2, \dots, x_N)$ as in $F_N(y_1, y_2, \dots, y_N)$.

The *Arrow impossibility theorem* says now that if Y is finite with at least three alternatives, then the restrictions that constitute the model are *incompatible*.[1]

Other classical models in the combinatorial approach were introduced by Gibbard and Satterthwaite in the 1970's, and also lead to impossibility results.[3,4]

2.2. *Topological models the crisp setting*

As mentioned in the introduction, in the 1980's Chichilnisky and others considered alternative models using a *topological* approach also in the crisp setting.[5–7]

Of course, the main underlying idea when launching these new alternative models was the search for possibility results in social choice.

But there is also a set of important *restrictive* facts that arise in the Arrovian model that were taken into account in order to build a not-so-restrictive alternative model, namely:

- The Arrovian model considers a *finite* set of alternatives.
- In the Arrovian model each individual has to define her preferences as a *ranking* on the set of alternatives.
- In the Arrovian model, each individual ranking must be a *total order*.

Remark 1. Despite it could perhaps seem a bit unfair towards the Arrovian view to avoid his assumption of a *total order* for each one of the individual preferences as well as for the aggregated preference, we should think that after the launching of the main Arrovian model and its impossibility result, a *search for possibility results* is in order in this theory, in the framework of Social Choice. Actually, the Arrovian impossibility theorem states that if Y is finite with at least three alternatives, then the concurrence of the restrictions C1, C2, C3 and C5 that constitute the model, provokes the negation of condition C4, so that the aggregation preference becomes *"dictatorial"*.

In the analysis of alternative models, there are also typical situations in Social Choice in which a kind of *sequential or serial dictatorship* appears. Basically, they are associated to Arrovian-like models where the preferences are total preorders instead of total orders, thus allowing ties.[19–22] However, the apparition of the so-called serial dictatorship is also considered an *impossibility* result. This fact contributed to additional search with Arrovian-like models of a different kind, namely models where *an infinite number of agents or decision-makers* is involved.[10,23–25]

At this point it is of particular importance to say that some of the last Arrovian-like models on which an infinite number of agents appear, use *topological tools* based on concepts like ultrafilters or ultraproducts. In addition, some of them leads to *possibility* results (in some sense) due to the absence of a (single) dictator, in spite of the new concept of an *invisible dictator*, that is introduced in these models. (For further details, consult the references, and in particular the key book by Kirman and Sonderman quoted there[10]).

Thus, these new Arrovian-like models perhaps suggested to other authors the use of *topology* in order to build alternative models to avoid social choice paradoxes. Moreover, although the original Arrovian model dealt only with total orders, the subsequent models also suggested not to be so restrictive, so allowing the consideration of other kinds of orderings as total preorders.

In a certain sense, perhaps the apparition of the Chichilnisky topological models could be (so-to-say) *expected or predictible at this stage*, and some Arrovian-like models with infinite agents could be considered in this literature as a bridge between the original Arrovian model and the new setting launched by Chichilnisky.

Unlike the (original) Arrovian model, in the Chichilnisky topological model we could be considering infinite sets of alternatives, the individual rankings of preferences could fail to be total orders, being instead any other possible kind of orderings, perhaps allowing ties. Moreover, it is not strictly necessary to know the whole ranking that each individual defines. Indeed, in some Chichilnisky model it is enough to work with the best element for each individual, disregarding the individual ranking.

Let us explain in detail these features: the set X may or may not be interpreted as a set of preferences defined as total orders on a set of alternatives. Instead, it could be interpreted as a set of individual orderings that may fail to be total orders. Sometimes it could also be interpreted as a set of *final choices that each individual makes* by selecting her best element.

Obviously, all this seems to be too ambiguous at first glance: *What is really the set X?*

The set X in a Chichilnisky model is just a *topological space*. It is irrelevant if each element $x \in X$ is interpreted as an individual preference defined through a total order on a set of alternatives, or if it is a different kind of ordering on the set of alternatives, or even if it is just the best alternative that an individual has selected as her best option.

All this will depend on the particular context.

Perhaps the most remarkable feature to be pointed out here is that Chichilnisky models furnish a new theoretical and *totally abstract* setting on which, depending on the context, either preference orderings or best individual alternatives, are understood as *points of a topological space*.

Let us see an example, that involves best choices instead of total orders on a set of alternatives:

<<A man is lost in the desert, and needs water. He does not know where the water is but has been told that there is an oasis not far from his position. He decides to walk always in the same direction in the hope of finally finding the oasis to survive. Thus, the set X of possible best choices here can be interpreted as the points of a circle that the magnetic needle of a compass could indicate>>.

That is, in this example we may identify X with the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, endowed with the restriction to S^1 of usual Euclidean topology of the plane $\mathbb{R}^2$.

In the *topological Chichilnisky model* X is a nonempty set endowed with a topology τ, and a social N-rule is a map $F_N : X^N \to X$ satisfying the following restrictions:[7]

(T1) *Continuity*: The N-rule $F_N : X^N \to X$ must be continuous (where X^N is endowed with the product topology).
(T2) *Unanimity*: For every $x \in X$ it holds that $F_N(x, \ldots, x) = x$.
(T3) *Neutrality or respect of anonimity*: $F_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = F_N(x_1, \ldots, x_N)$ for every rearrangement σ of the set $\{1, \ldots, N\}$ and every $(x_1, x_2, \ldots, x_N) \in X^N$.

It is important to point out here that the set X may now be *infinite*.

2.3. *Algebraical models in the crisp setting*

As in the topological case, the *algebraical models* are totally *abstract* and were introduced to get rid of social choice paradoxes. The former papers in this approach appeared in the 1990's.[8,9,26] By the way, the consideration of preferences defined on sets endowed with some algebraical structure is typical in Utility Theory.[27–30] (See also Secs. 4.5 and 4.6 in the classical book by Bridges and Mehta on representation of preferences[31]).

Again, in these new *algebraical* models the set X may be interpreted in many ways, depending on the context.

Now X is also a nonempty set, endowed with a binary operation $\bar{+}$. Usually this binary operation $\bar{+}$ is asked to accomplish certain conditions (i.e.: associativity or commutativity), so that $(X, \bar{+})$ could have some typical algebraical structure (e.g.: semigroup, monoid or group).

Once more it is important to point out that X may possibly be *infinite*.

Again an example of this situation could be $X = S^1$, but now identifying the unit circle to the set of complex numbers of modulus 1, that is $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, that endowed with the usual multiplication of complex numbers $(\cdot)$ becomes a group.

Let us see another easier example in this direction:

<<A man usually goes to the market twice a week. One day, he decides to buy a basket $(x_1, x_2, \ldots, x_k)$ of k goods, and the next day in the same week he buys a

different basket $(y_1, y_2, \ldots, y_k)$. *The next week, however, he has not enough time to go twice to the market, so that he decides to buy at once the basket* $(z_1, z_2, \ldots, z_k)$, *where* $z_i = x_i + y_i (i \in \{1, \ldots, k\})$. >>

In this latter example, X can be identified to $\mathbb{R}^k$, and endowed with the coordinatewise sum also becomes a group.

In these algebraical models, a social N-rule is a map $F_N : X^N \to X$ satisfying the following restrictions:

(A1) *Homomorphism or respect of the algebraical operation*: $F_N(x_1 \bar{+} y_1, \ldots, x_N \bar{+} y_N) = F_N(x_1, \ldots, x_N) \bar{+} F_N(y_1, \ldots, y_N)$, for every $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N) \in X^N$.
(A2) *Unanimity.*
(A3) *Neutrality.*

3. Functional Equations Related to Social Choice: The Crisp Setting

Looking at the latter models introduced in the crisp setting, we may notice that in most cases the search for aggregation rules leads in a natural way to *functional equations.*

3.1. *From combinatorial to topological and algebraical models. A motivation based on functional equations*

We know that the Arrovian model leads to the Arrow's impossibility theorem.[1] In addition, other combinatorial models also lead to social choice paradoxes.[3,4]

Thus, it is important to look for alternative approaches in order to solve social choice paradoxes (i.e. generating *possibility* results). As mentioned in the Introduction, this is the case of several topological and algebraical models.

At this stage, it is important to point out that the introduction in the literature of those alternative models (topological and algebraical) may be *motivated in terms of functional equations.* As a matter of fact, the complexity of a good notation to express the conditions involved in the Arrovian model by means of functional equations, could lead us to consider *alternative* combinatorial models.

Thus, we could consider a (still combinatorial) model consisting only of the following conditions on an *abstract* nonempty set X, to which we could give various interpretations, depending on the context, just as in the topological and algebraical models introduced above.

(i) *Whole domain*
(ii) *Rank restriction*
(iii) *Unanimity*
(iv) *Neutrality.*

Consequently, we will look for maps $F_N : X^N \to X$ that solve simultaneously both the *functional equations* of unanimity and neutrality. This leads to the key concept of a general N-mean that we introduce in the next definition.

Definition 1. Let X be a nonempty set. Let $N \geq 2, N \in \mathbb{N}$. A*general N-mean* is a map $F_N : X^N \to X$ that simultaneously satisfies the functional equations of unanimity and neutrality. For $N = 2$, general 2-means are also called *bivariate means*.

This concept will be studied in Sec. 4.

Obviously, the next step is just adding some extra conditions to a general N-mean. The extra conditions should be related to the additional structure on the set X involved. Thus, when X is endowed with a topology τ and we look for a *continuous* general N-mean $F_N : X^N \to X$ we reach the topological Chichilnisky model. Similarly, if X is endowed with a binary operation $\bar{+}$ and we look for a general N-mean that is also an homomorphism, we get an algebraical model.

3.2. *Functional equations in the topological models*

Definition 2. Given a topological space (i.e.: a nonempty set X endowed with a topology τ), a (topological) social aggregation N-rule is also called a *topological N-mean*, that is, a *continuous* map $F : X^N \to X$ that solves simultaneously both involved *functional equations* of:

1. *Unanimity*: $F_N(x, \ldots, x) = x$, for every $x \in X$.
2. *Neutrality*: $F_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = F_N(x_1, \ldots, x_N)$ for every rearrangement σ of the set $\{1, \ldots, N\}$ and every $(x_1, x_2, \ldots, x_N) \in X^N$.

At this point we may observe that any general N-mean on a nonempty set X can still be considered as a *topological* N-mean just endowing X with the discrete topology.

The question of existence of topological N-means was already analyzed and solved in several contexts that are based on old problems arising in *algebraical topology*, where the possibility of defining a general mean furnishes important information about the intrinsic topological structure of the space considered.[32]

The merit of recent research on the mathematical theory of social choice is realizing that those purely abstract results arising in Topology could indeed be applied or reinterpreted to prove deep results on the aggregation of individual preferences or choices, leading to possibility results (i.e., avoiding social choice paradoxes) in some particular cases.

In order to quote now some of the main important results in this direction, let us introduce some definitions.

Definition 3. A *cellular complex* (or CW-complex) is a topological space that can be constructed inductively by adding n-cells. An *n-cell* is homeomorphic to the n-dimensional unit open ball. Cellular complexes include a wide class of spaces, namely Euclidean spaces, all manifolds and polyhedra.

A cellular complex is said to be *parafinite* if, for every $n \in \mathbb{N}$, there are only a finite number of n-cells. (Examples are spheres, tori, balls, and cubes). Parafinite cellular complexes may be infinite dimensional, as they may contain an infinite number of cells, although they may contain only a finite number of cells in each dimension.

Every parafinite cellular complex can be regarded as a topological subset of $\mathbb{R}^\infty$, the countably infinite-dimensional Euclidean space.

Definition 4. A topogical space (X, τ) is *contractible* if it is continuously deformable into one of its points. (For further details, see pp 61-97 of the book by Rohlin and Fuchs mentioned in the bibliography.[33])

That is, if (X, τ) is contractible there exists a continuous map $H : X \times [0, 1] \longrightarrow X$ such that $H(x, 0) = x; H(x, 1) = x_0$, for every $x \in X$, with $x_0 \in X$ fixed.[33]

The following important result was essentially proved by Aumann.[32] It was reobtained and adapted to the social choice framework by Chichilnisky:[7]

Theorem 1. *Let (X, τ) be a connected parafinite cellular complex. Then there exists a topological N-mean $F_N : X^N \longrightarrow X$ for every $N \geq 2, N \in \mathbb{N}$ if and only if the space (X, τ) is contractible.*

(In other words, when (X, τ) is a connected parafinite cellular space, the Chichilnisky topological model leads to a *possibility* result for every $N \geq 2, N \in \mathbb{N}$ if and only if (X, τ) is contractible. Equivalently, it leads to a social choice paradox or impossibility result for some $N \geq 2, N \in \mathbb{N}$ if and only if (X, τ) is not contractible).

3.3. *Functional equations in the algebraical models*

Definition 5. Let X be a nonempty set and $\bar{+}$ a binary operation defined on X. An (algebraical) social aggregation rule is also called an *algebraical N-mean*, that is, a map $F : X^N \to X$ that solves simultaneously the *functional equations* of:

1. *Unanimity*: $F_N(x, \ldots, x) = x$, for every $x \in X$.
2. *Neutrality*: $F_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = F_N(x_1, \ldots, x_N)$ for every rearrangement σ of the set $\{1, \ldots, N\}$ and every $(x_1, x_2, \ldots, x_N) \in X^N$.
3. *Homomorphism*: It holds that $F_N(x_1, \ldots, x_N) \,\bar{+}\, F_N(y_1, \ldots, y_N) = F_N(x_1 \bar{+} y_1, \ldots, x_N \bar{+} y_N)$, for every $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N) \in X^N$.

Remark 2. When a set X is endowed with both a topology τ and a binary operation $\bar{+}$, an N-rule could be, simultaneously, a topological and algebraical N-mean. An obvious *example* is the arithmetic mean $F_N(a_1, \ldots, a_N) = \frac{a_1 + \ldots + a_n}{n}$ $((a_1, \ldots, a_N) \in \mathbb{R}^N)$, where the real line $\mathbb{R}$ is endowed with the usual addition $+$ and the Euclidean topology.

Let us see now which is the situation concerning the existence of *algebraical N-means*. As in the case of topological N-means, it will strongly depend on the additional (now *algebraical*) structure involved.

We shall start with the case in which the considered space is a *group*. The reason for paying a particular attention to groups, leaving apart their own interest by themselves, is also the connection between the existence of topological N-means on connected topological spaces and the existence of algebraical N-means on their homotopy groups. (See Subsec. 3.4 below).

Definition 6. Let $(G, \bar{+})$ be a group. An element $g \in G$ is said to be *nilpotent* or a *torsion element* if there exists some $N \geq 2, N \in \mathbb{N}$ such that $N \cdot g = e$ and $g \neq e$, where $N \cdot g = g\bar{+} \dots (\mathrm{N}-times) \dots \bar{+}g$ and e denotes the unit element of G with respect to the group operation $\bar{+}$. The group $(G, \bar{+})$ is said to be *abelian* if $x\bar{+}y = y\bar{+}x$ for every $x, y \in X$. It is said to be *torsion free* if for every $g \in G$, g is not a nilpotent element.

The group $(G, \bar{+})$ is said to be *divisible* if for every $N \geq 2, N \in \mathbb{N}$, and every $g \in G$ there exists an element $h \in G$ (that may or may not be unique) such that $N \cdot h = g$. If given $g \in G$ there exists a *unique* $h \in G$ such that $N \cdot h = g$, we shall also use the notation $h = \frac{g}{N}$.

Lemma 1. *Let $(G, \bar{+})$ be a group. Let $N \geq 2, N \in \mathbb{N}$. Suppose that there exists an algebraical N-mean $F_N : G^N \to G$. Then $(G, \bar{+})$ is abelian and for each $g \in G$ there exists a unique $h \in G$ such that $N \cdot h = g$. Moreover, the N-mean F_N is unique, and it is defined as the convex mean:*

$$F_N(g_1, \dots, g_N) = \frac{g_1\bar{+} \dots \bar{+}g_N}{N}, \ (g_1, \dots, g_N \in G).$$

Proof. This is already known. (See Theorem 1 in the key reference.[26] Complete with some subsequent results[8]). ☐

Theorem 2. *Let $(G, \bar{+})$ be a group. Then the following statements are equivalent:*

(i) *For every $N \geq 2, N \in \mathbb{N}$ there exists a unique algebraical N-mean on G.*
(ii) *The group $(G, \bar{+})$ is abelian, torsion free and divisible.*
(iii) *The group $(G, \bar{+})$ is a vector space over the scalar field $\mathbb{Q}$ of rational numbers.*
(iv) *The group $(G, \bar{+})$ is isomorphic to a direct sum of copies of the additive group of rational numbers $(\mathbb{Q}, +)$.*

Moreover for each $N \geq 2, N \in \mathbb{N}$, the unique algebraical N-mean is given by the convex mean $F_N(g_1, \dots, g_N) = \frac{g_1\bar{+}\dots\bar{+}g_N}{N}$, $(g_1, \dots, g_N \in G)$.

(In particular, if $(G, \bar{+})$ is a group the algebraical model leads to a possibility result for every $N \geq 2, N \in \mathbb{N}$ if and only if $(G, \bar{+})$ is abelian, torsion free and divisible).

Proof. This is well-known. (See p. 10 in the classical book by Kaplansky.[34] Complete with other results on algebraical aggregation[8]). ☐

Although the interest of algebraical *groups* in the social choice framework is undeniable, in our opinion there are many contexts in which the structure of a group is *too restrictive*. Instead, the use of a *semigroup* (i.e.: a nonempty set X endowed with an *associative* binary operation $\bar{+}$) seems to be much more adequate. A very important class of structures that can be considered as semigroups in a natural way is that of *totally ordered sets*. Indeed, if X is a nonempty set endowed with a total order $\preceq$ (i.e.: $\preceq$ is a reflexive, antisymmetric, transitive and total binary relation defined on X), the set X becomes a semigroup through the latticial operation $\vee$ given by $x \vee y = max\{x, y\}$ $(x, y \in X)$, where the maximum is taken with respect to the total order $\preceq$ defined on X.

Indeed, the class of *semilattices* (instead of, just, totally ordered sets, that is more restrictive) can also be considered as a subclass of the class of semigroups.

Definition 7. Let X be a nonempty set, and $\preceq$ a partial order (i.e.: a reflexive, antisymmetric and transitive binary relation) defined on X. Then the pair $(X, \preceq)$ is said to be a *lattice* if for every pair of elements $x, y \in X$ there exist both a greatest lower bound $z \in X$ and a lowest upper bound $t \in X$ with respect to $\preceq$. That is, for every $a \in X$ such that $a \preceq x$ and $a \preceq y$ it holds that $a \preceq z$, and also for every $b \in X$ such that $x \preceq b$ and $y \preceq b$ it holds that $t \preceq b$. The greatest lower bound of x and y is called *meet* and it is denoted by $x \wedge y$. The lowest upper bound of x and y is called *join* and it is denoted by $x \vee y$. (For more details on Lattice Theory, the classical book by Birkhoff is a suitable reference[35]).

Definition 8. Let X be a nonempty set, and $\preceq$ a partial order defined on X. Then the pair $(X, \preceq)$ is said to be a *semilattice* if for every pair of elements $x, y \in X$ there exists a lowest upper bound $t \in X$, that we shall denote $x \vee y$.

Remark 3. If $(X, \preceq)$ is a lattice, then for every $N \geq 2, N \in \mathbb{N}$, the map $F_N : (X, \wedge)^N \longrightarrow (X, \wedge)$ defined by $F_N(x_1, \ldots x_N) = x_1 \wedge x_2 \wedge \ldots \wedge x_N, (x_1, x_2, \ldots, x_n) \in X^N$ is plainly an *algebraical N-mean* for the semigroup structure $(X, \wedge)$. Similarly, if $(X, \preceq)$ is a semilattice, the map $G_N : (X, \vee)^N \longrightarrow (X, \vee)$ defined by $G_N(x_1, \ldots x_N) = x_1 \vee x_2 \vee \ldots \vee x_N, (x_1, x_2, \ldots, x_n) \in X^N$ is an *algebraical N-mean* for the semigroup structure $(X, \vee)$.

3.4. *Algebraical models controlling topological models*

There is a suggestive relationship between topological and algebraical models.

In some way, *algebraical models control topological models*. Let us see why: Let (X, τ) be a connected Hausdorff topological space. Notice that a *continuous* map $F_N : X^N \to X$ immediately induces group homomorphisms $(\phi_r)_N : \pi_r(X^N) \to \pi_r(X)(r \in \mathbb{N})$ on the homotopy groups of the spaces X^N and X. (See p. 361 in the reference[33]). But $\pi_r(X^N)$ is isomorphic to $(\pi_r(X))^N$. From the functoriality of those correspondences mapping topological spaces into their homotopy groups, it is straightforward to see that if F_N is unanimous and neutral, the same happens to $(\phi_r)_N$. Consequently, we obtain the following key result:[8]

Proposition 1. *The existence of a topological N-mean on a connected Hausdorff topological space immediately induces the existence of algebraical N-means on each of its homotopy groups.*

4. Functional Equations Related to Aggregation Rules: The Means

In the general definition of an *N-mean*, topology and algebra play no role (*a priori*).

It is crucial to analyze this general case (without topological or algebraical restrictions), *looking for solutions of the simultaneous equations that define a general N-mean.*

Obviously, if we have at hand a list of possible solutions of such functional equations, in the topological models we will only need to select the continuous ones among them, whereas in the algebraical models we should select those general N-means that are also algebraical homomorphisms.

Moreover, we will realize which are the reasons for the non existence of a topological or algebraical N-mean, observing that *the lack of solutions is usually related to the extra* (topological or algebraical) *structure involved*, because of the the following key result.

Theorem 3. *Let X be a nonempty set. Then for every $N \geq 2, N \in \mathbb{N}$ there exists a general N-mean $F_N : X^N \to X$.*

Proof. Using if necessary the well-ordering axiom of Set Theory, we may assume that X is endowed with a total order, say $\preceq$. It is clear that the map $F_N : X^N \to X$ defined by $F_N(x_1, \ldots, x_N) = max\{x_1, \ldots x_N\}$ for every $(x_1, \ldots, x_N) \in X^N$, where the maximum is taken with respect to the given total order $\preceq$, is indeed a general N-mean. $\square$

The study of general N-means that accomplish additional properties can be used in the topological or algebraical social choice frameworks, in the search for suitable aggregation rules. As a matter of fact, the *functional equations* that define a general N-mean have been studied in several other contexts.[18,36–38]

5. Functional Equations Related to Aggregation Rules: Associativity

5.1. *Associative maps*

Definition 9. Let X be a nonempty set. A map $F : X^2 \to X$ is said to be *associative* if $F(F(x,y),z) = F(x, F(y,z))$ for every $x, y, z \in X$.

We may immediately observe that an associative map F endows X with an algebraic structure of *semigroup* $(X, \bar{+})$ where $\bar{+}$ is the binary operation defined by $x \bar{+} y = F(x,y) (x, y \in X)$.

(In other words, the *solutions of the functional equation of associativity* are exactly the binary operations that define on X a structure of a *semigroup*).

Remark 4. *Associative maps can always be defined on any nonempty set X.* An obvious example is the *constant* map $F : X^2 \to X$ given by $F(x, y) = a$ for every $x, y \in X$ where a is an element of X fixed a priori.

Another easy example is the map $F : X^2 \to X$ given by $F(x, y) = max\{x, y\}$ for every $x, y \in X$, where the maximum is taken with respect to a total order $\preceq$ defined on X, as in Theorem 3 above.

A third clear example is a *projection* map, as $F(x, y) = x$ for every $x, y \in X$ (or, similarly $F(x, y) = y$ for every $x, y \in X$).

(For a further account on the study of the associativity equation and its generalizations, as well as a large variety of applications in different contexts, consult the seminal and normative work made by János Aczél[39–41]).

5.2. *A motivation coming from social choice*

As mentioned in the introduction, the typical problem in social choice is the construction of a method to aggregate the preferences of no matter which (finite) *number of agents*. We should observe here that, even when social aggregation N-rules are available for every $N \geq 2, N \in \mathbb{N}$, it could happen that *there is no relationship between, say, the existing N-rule and the existing M-rule if $M \neq N$*. This fact could carry some extra difficulties:

<<Suppose that in a poll a finite number of persons have expressed their preferences. Once the scrutiny starts, it is desirable to have an algorithm that, gradually, furnishes the social preference that results from the aggregation of the first $2, 3, 4, \ldots, N$ ballots. Of course, it would be interesting here to have an algorithm that uses the aggregated social rule of the first K individual ballots, jointly with the $(K + 1)$-th individual one, to compute the aggregated social preference of such $K + 1$ individuals, without re-starting all the calculations each time that a new vote is scrutinized.>>

The underlying mathematical problem is the following:

Is it possible to modify a bivariate mean to get an N-mean for any $N > 2, N \in \mathbb{N}$?

This question is very important, because it is related to a general question studied and analyzed since long, namely the convenience or not of supporting social choice only on *binary* comparisons or operations.[42] Observe that if the question introduced above concerning N-means has a positive answer, the aggregation of rules for any number of agents will be based only on the aggregations made for two agents, or *binary* aggregation.

Sometimes the aforementioned mathematical problem has a *positive* solution. For instance, this would happen if an *associative* bivariate mean exists.

5.3. *Associative bivariate means*

As a consequence of Remark 4, we can say that *the solutions of the functional equation of associativity only deserve interest if some other additional properties or restrictions are involved*. This study of special solutions of the associativity equations in various different contexts is a common item in this literature.[43–46]

A key particular case to be considered in this Subsec. 5.3 concerns solutions of the associativity equation *that are bivariate means as well*, since they provoke the existence of N-means for every $N \geq 2, N \in \mathbb{N}$. These are called *associative bivariate means*. An associative bivariate mean $F_2 : X^2 \to X$ on a nonempty set X furnishes X with a *semigroup* operation $\bar{+}$ (with $x\bar{+}y = F_2(x,y),\ x,y \in X$) satisfying the following conditions:

(i) *Associativity*: $(x\bar{+}y)\bar{+}z = x\bar{+}(y\bar{+}z)$ $(x,y,z \in X)$.
(ii) *Commutativity*: $x\bar{+}y = y\bar{+}x$ $(x,y \in X)$.
(iii) *Idempotence*: $x\bar{+}x = x$ $(x \in X)$.

Observe that commutativity reflects the neutrality of F_2, whereas idempotence reflects its unanimity.

Remark 5. Given a nonempty set X, a bivariate mean defined on X may fail to be associative, in general. An *example* is the convex bivariate mean on the real line $\mathbb{R}$, given by $F_2(x,y) = \frac{x+y}{2}$ $((x,y) \in \mathbb{R}^2)$. Notice that $F_2(F_2(1,0),0) = \frac{1}{4} \neq \frac{1}{2} = F_2(1, F_2(0,0))$.

The following straightforward result is now in order.

Proposition 2. *Let X be a nonempty set, and $F : X^2 \to X$ an associative bivariate mean defined on X. Given $N > 2, N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_N \in X$, define recurrently: $F_3(x_1,x_2,x_3) = F(F(x_1,x_2),x_3)$ and, in general, $F_{K+1}(x_1,x_2,\ldots,x_{K+1}) = F(F_K(x_1,x_2,\ldots,x_K),x_{K+1})$. Then the map $F_N : X^N \to X$ is an N-mean on X.*

Next result reinforces Remark 4 as well as the particular case of Theorem 3 for $N = 2$.

Proposition 3. *Let X be a nonempty set. Then there exists an associative bivariate mean $F : X^2 \to X$.*

Proof. There is no loss of generality in assumming that X is endowed with an ordering $\preceq$ such that $(X, \preceq)$ is a semilattice. (If necessary we could use the *well ordering axiom* of Set Theory in order to consider X endowed with a well ordering, that is a particular case of total order, hence it defines a lattice. Anyway, without making use of that axiom, we can consider the following *trivial* construction: Fix $a \in X$ and define $\preceq$ by declaring only that $x \preceq x$ and $x \preceq a$ for every $x \in X$). Let $\vee$ denote the corresponding join operation. Then the map $F : X^2 \to X$ given by $F(x,y) = x \vee y$ $(x,y \in X)$ is an associative bivariate mean. $\square$

The kind of associative bivariate mean used in the proof of Proposition 3 is actually *universal* as next Theorem 4 shows.

The key theorem

Theorem 4. *Let X be a nonempty set. Let $F : X^2 \to X$ be an associative bivariate mean defined on X. Then there exists a partial order $\preceq$ defined on X such that $(X, \preceq)$ is a semilattice and $F(x,y) = x \vee y$ $(x, y \in X)$, where $\vee$ stands for the join operation on $(X, \preceq)$.*

Proof. Define $\preceq$ on X as follows: $x \preceq y \iff F(x,y) = y$ $(x, y \in X)$. The binary relation $\preceq$ is reflexive by the unanimity property of F. It is antisymmetric, due to the commutativity of F. Let us see that $\preceq$ is also transitive: If $x \preceq y$ and $y \preceq z$, we have that $F(x,y) = y$ and $F(y,z) = z$. Thus, by associativity of F it follows that $F(x,z) = F(x, F(y,z)) = F(F(x,y), z) = F(y,z) = z$. Therefore $x \preceq z$. Let us see now that $F(x,y)$ is the lowest upper bound of x and y with respect to $\preceq$, so that $(X, \preceq)$ becomes a semilattice: Indeed, $x \preceq F(x,y)$ because $F(x, F(x,y)) = F(F(x,x), y) = F(x,y)$. Similarly, $y \preceq F(x,y)$ because $F(y, F(x,y))$ =F(F(x,y),y) = = F(x, F(y,y)) = F(x,y). Finally, if $x \preceq a$ and $y \preceq a$ for some $a \in X$, we have that $F(x,a) = a = F(y,a)$. Hence, by associativity of F it follows that $F(F(x,y), a) = F(x, F(y,a)) = F(x,a) = a$. Therefore $F(x,y) \preceq a$. Consequently $F(x,y) = x \vee y$ $(x, y \in X)$ and we are done. $\square$

5.4. *Theoretical consequences of the characterization of associative bivariate means*

The results in the previous Subsec. 5.3 carry important theoretical consequences in topological and algebraical contexts. Proposition 2 states that an associative bivariate mean induces general N-means for every $N \geq 2, N \in \mathbb{N}$. Moreover, an associative bivariate mean can be interpreted through Proposition 3 and Theorem 4 as the join operation of a semilatticial partial order defined on X.

Definition 10. Let (X, τ) be a topological space. Let $\preceq$ be a semilatticial partial order defined on X. Let $\vee$ denote the corresponding join operation. The partial order $\preceq$ is said to be *τ-continuous* if the associated bivariate mean $F : X^2 \to X$ defined by $F(x,y) = x \vee y (x, y \in X)$ is continuous (considering on X^2 the product topology $\tau \times \tau$ and on X the given topology τ).

Theorem 5. *Let (X, τ) be a non-contractible connected parafinite cellular complex. Then, any semilatticial partial order $\preceq$ defined on X is τ-discontinuous.*

Proof. A semilatticial partial order $\preceq$ can always be defined on X, by Proposition 3. However, by Proposition 2, if $\preceq$ is τ-continuous then there exists a *topological* N-mean on (X, τ), for every $N \geq 2, N \in \mathbb{N}$. This fact contradicts Theorem 1, so that $\preceq$ must be a fortiori τ-discontinuous. $\square$

A similar result appears in the algebraical context.

Definition 11. Let $(G, \bar{+})$ be a group. Let $\preceq$ be a semilatticial partial order defined on X. Let $\vee$ denote the corresponding join operation. The partial order $\preceq$ is said to be $\bar{+}$-*compatible* if $(g_1 \bar{+} g_2) \vee (g_3 \bar{+} g_4) = (g_1 \vee g_3) \bar{+} (g_2 \vee g_4)$ for every $g_1, g_2, g_3, g_4 \in G$.

Theorem 6. *Let $(G, \bar{+})$ be a group. If there exists a $\bar{+}$-compatible semilatticial partial order $\preceq$ defined on G , then $(G, \bar{+})$ is abelian, torsion free and divisible.*

Proof. It is an straightforward consequence of Proposition 2, Lemma 1 and Theorem 2. □

6. Functional Equations Related to Aggregation Rules: The Fuzzy Setting

In several contexts arising in social choice it is usual to consider a *fuzzy* setting to deal with preferences and aggregation rules. In this section we shall analyze some functional equations that correspond to this fuzzy setting.

First we recall a basic definition.

Definition 12. Let U be a nonempty set (henceforward called *universe*). Let $(\mathcal{L}, \preceq)$ be a lattice that has a greatest element $\bar{1}$ and a smallest element $\bar{0}$. Then, a *fuzzy set* is a function (usually called the *membership* function) $A : U \longrightarrow \mathcal{L}$. The *support* of A is the set $Supp(A) = \{x \in U : A(x) \neq \bar{0}\}$, and the *kernel* of the fuzzy set A is the set $KerA = \{x \in U : A(x) = \bar{1}\}$. A fuzzy set A is said to be *normal* if it has nonempty kernel.

6.1. *Fuzzy aggregation of preferences*

There are many possible fuzzy approaches to aggregate preferences.

Among them, we shall consider the following one:

Given a nonempty set of alternatives U, the preferences of each individual of a (finite) society are represented by a fuzzy set $A : U \longrightarrow \mathcal{L}$, where $(\mathcal{L}, \preceq)$ is the corresponding lattice, in a way that the kernel $KerA$ represents the elements that for such individual are the most preferred, and the membership function A gives the idea of a ranking or degree of membership of the elements of the universe U to the fuzzy set of preferred elements of the given individual.

Two *examples* that correspond to this approach are the following:

1. A set U of actors have made a casting to take part in a movie. The examiner qualifies each of them with a linguistic mark as $P =$ "poor", $G =$ "good" or $E =$"excellent". The set $\mathcal{L} = \{P, G, E\}$ is totally ordered in the natural way.
2. The individuals of the set U of employees of a firm are classified by the different tasks that each of them is able to carry out. If $\mathcal{T}$ denotes the set of different

tasks, its power set $P(\mathcal{T})$ is partially ordered by set-inclusion ($\subseteq$), in the obvious way. Indeed, this partial order $\subseteq$ is latticial, and it has a smallest element $\emptyset$ and a greatest element $\mathcal{T}$. Here it is defined a map $A : U \to P(\mathcal{T})$ that assigns to each individual the set of tasks that such person is able to accomplish.

Let now $N \geq 2, N \in \mathbb{N}$ and suppose that the N individuals of a society have defined their preferences on the set U, and that all the individuals use the same lattice $(\mathcal{L}, \preceq)$ to build them. Thus, we have at hand N fuzzy sets $A_1, A_2, \ldots, A_N : U \to \mathcal{L}$. In this setting, we understand a *social aggregation N-rule* as a map $F_N : \mathcal{F}^N \to \mathcal{F}$, where $\mathcal{F}$ denotes the family of fuzzy sets whose universe is U and whose corresponding lattice is $(\mathcal{L}, \preceq)$. Of course, it is natural to impose some conditions on F_N, as the neutrality and the respect of the unanimity defined in the obvious way. When both U and $\mathcal{L}$ are topological spaces it is also usual to ask $A_1, \ldots, A_N$ and $F_N(A_1, \ldots, A_N)$ to be continuous maps from U to $\mathcal{L}$. Thus, the rule F_N is said to be *continuous* if $F_N(A_1, \ldots, A_N) : U \to \mathcal{L}$ is a continuous map provided that $A_1, \ldots, A_N : U \to \mathcal{L}$ are all continuous.

Again, the existence of an *associative* neutral and unanimous 2-rule (or *bivariate rule*) immediately gives rise to the existence of a neutral and unanimous N-rule, for every $N \geq 2,\ N \in \mathbb{N}$. If in addition the associative bivariate rule is continuous, then the corresponding N-rule is also continuous.

Observe that from a purely abstract point of view, this approach is totally similar to other ones already considered in Secs. 3 and 4, in which we were looking for general N-means and associative bivariate means on a nonempty set X: the main set here is now $\mathcal{F}$, the family of fuzzy sets whose universe is U and whose corresponding lattice is $(\mathcal{L}, \preceq)$.

Indeed, general means or associative maps on the underlying lattice $\mathcal{L}$ immediately induce aggregation rules or associative maps on $\mathcal{F}$, as next Proposition 4 shows.

Proposition 4.

(i) *For any $N \geq 2, N \in \mathbb{N}$, every general N-mean on $\mathcal{L}$ induces a neutral and unanimous social aggregation N-rule on $\mathcal{F}$.*

(ii) *Every associative map $f : \mathcal{L}^2 \to \mathcal{L}$ induces an associative map $F : \mathcal{F}^2 \to \mathcal{F}$.*

Proof. We will only prove part (i). The proof of part (ii) is similar. Let $f_N : \mathcal{L}^N \to \mathcal{L}$ be a general N-mean on $\mathcal{L}$. Define now $F_N : \mathcal{F}^N \to \mathcal{F}$ as follows: Given N fuzzy sets $A_1, A_2, \ldots, A_N : U \to \mathcal{L}$, let $F_N(A_1, \ldots, A_N) : U \longrightarrow \mathcal{L}$ be the map defined by $F_N(A_1, \ldots, A_N)(x) = f_N(A_1(x), \ldots, A_N(x))$ $(x \in U)$. It is straightforward to check that F_N is actually neutral and unanimous. $\square$

Remark 6. In many fuzzy contexts, if $(\mathcal{L}, \preceq)$ is the corresponding lattice of a fuzzy set, a map $f : \mathcal{L}^2 \to \mathcal{L}$ is said to be an *aggregation operator*. The underlying idea is very simple. If we consider two fuzzy sets that share the same universe $\mathcal{U}$ and lattice $\mathcal{L}$, say $A_1, A_2 : \mathcal{U} \to \mathcal{L}$, using the operator f a new *aggregated* fuzzy set $A_{f(A_1,A_2)}$ is given by the construction $A_{f(A_1,A_2)}(u) = f(A_1(u), A_2(u))$ for every element u in the universe $\mathcal{U}$. Among the aggregation operators, a very important class of fuzzy sets is that of *associative* maps.[47] In our context of aggregation of preferences, a key subclass of associative aggregation operators is that of *associative bivariate means*, now considered in the fuzzy setting.

As a direct consequence of Proposition 2 and Proposition 4, we observe that in this fuzzy setting an associative bivariate mean generates a *possibility theorem*. Therefore, the aggregation operators that are associative bivariate means will be of capital importance in this fuzzy contexts of aggregation of preferences. By Proposition 3 and Theorem 4, these aggegation operators can be identified to the semilatticial preorders that could be defined on the lattice $\mathcal{L}$.

It is important to point out that a semilatticial preorder (say, $\precsim$) defined on the corresponding lattice $(\mathcal{L}, \preceq)$ of a fuzzy set could be totally *independent* from $\preceq$. A priori, there is no relationship between $\precsim$ and $\preceq$.

Given a lattice structure $(\mathcal{L}, \preceq)$ it is an *open problem* to describe all the possible semilatticial partial orders that can be defined on $\mathcal{L}$. To this task, it may help us to know that in some cases a family of associative aggregation operators (of which we should select the ones that are also bivariate means) has been analyzed.[47]

6.2. *Associative bivariate means in the fuzzy setting vs. triangular norms and conorms*

In this fuzzy context, we may observe that the associative bivariate means defined on a lattice $(\mathcal{L}, \preceq)$ with a greatest element $\bar{1}$ and a smallest element $\bar{0}$ satisfy functional equations that are quite similar to that of triangular fuzzy norms and conorms. Fuzzy norms are conorms and very important tools, especially used in the fields of fuzzy logic, fuzzy arithmetic and expert systems.[48] (See e.g. Ch. 8 in the reference[48]).

Let us recall their definitions.

Definition 13. Let $(\mathcal{L}, \preceq)$ be a lattice that has a greatest element $\bar{1}$ and a smallest element $\bar{0}$. A map $T : \mathcal{L}^2 \to \mathcal{L}$ is said to be a *triangular norm* if it satisfies the following conditions:

(i) *Associativity*: $T(T(x,y),z) = T(x,T(y,z))$ $(x,y,z \in \mathcal{L})$.
(ii) *Commutativity*: $T(x,y) = T(y,x)$ $(x,y \in \mathcal{L})$.
(iii) *Monotonicity*: $T(a,x) \preceq T(b,x)$ for every $x,a,b \in \mathcal{L}$ such that $a \preceq b$.
(iv) *Boundary conditions*: $T(x,\bar{1}) = x; T(x,\bar{0}) = \bar{0}$ $(x \in \mathcal{L})$.

Similarly, a map $S : \mathcal{L}^2 \to \mathcal{L}$ is said to be a *triangular conorm* if it satisfies the conditions of *associativity*, *commutativity*, and *monotonicity*, as well as the following *boundary conditions*: $S(x,\bar{1}) = \bar{1}; S(x,\bar{0}) = x$ $(x \in \mathcal{L})$.

Although triangular norms and conorms are *associative* (and commutative) maps defined on $\mathcal{L}$, *in general they are not associative bivariate means*, because the elements $x \in \mathcal{L} \setminus \{\bar{0}, \bar{1}\}$ may fail to satisfy the *unanimity* condition $T(x,x) = x$ when T is a norm or $S(x,x) = x$ when S is a conorm.

Despite the almost universal use of triangular norms and conorms in many fuzzy contexts, from the point of view of *social choice* that we have adopted above their interest is limited by the following important fact:

Theorem 7. *Let $(\mathcal{L}, \preceq)$ be a lattice that has a greatest element $\bar{1}$ and a smallest element $\bar{0}$. The only associative bivariate mean that is a triangular norm (respectively, conorm) on $\mathcal{L}$ is the minimum $F(x,y) = x \wedge y = min\{x,y\}$ (respectively, the maximum $G(x,y) = x \vee y = max\{x,y\}$ $(x, y \in \mathcal{L})$.*

Proof. We furnish the proof for the case of a triangular norm. The situation for conorms is similar. Thus, let $F_2 : \mathcal{L}^2 \to \mathcal{L}$ be an associative bivariate mean that is also a triangular norm. Given $a, b \in \mathcal{L}$ such that $a \preceq b$, using the unanimity, monotonicity, commutativity and boundary conditions that satisfies F_2 we have that $a = F_2(a,a) \preceq F_2(b,a) = F_2(a,b) \preceq F_2(a,\bar{1}) = a$. Therefore $F_2(a,b) = F_2(b,a) = a = min\{a,b\}$. □

Remark 7.

(i) As we may expect, not every associative bivariate mean on a lattice $(\mathcal{L}, \preceq)$ with a greatest and a smallest element is a triangular norm or conorm. In other words, the associative bivariate means are not necessarily defined by the maximum or the minimum. An *example* is $\mathcal{L} = \{1,2,3\}$ endowed with the usual Euclidean order, and the associative bivariate mean $F_2 : \mathcal{L}^2 \to \mathcal{L}$ given by $F_2(1,1) = F_2(1,2) = F_2(1,3) = F_2(2,1) = F_2(3,1) = 1; F_2(2,2) = 2; F_2(2,3) = F_2(3,2) = F_2(3,3) = 3$.

(ii) Conversely, not every triangular norm or conorm on a lattice $(\mathcal{L}, \preceq)$ with a greatest and a smallest element is an associative bivariate mean. Classical *examples* in $\mathcal{L} = [0,1] \subset \mathbb{R}$ endowed with the usual Euclidean order $\leq$ are the *product norm* $P(x,y) = xy$, the *Lukasiewicz norm* $L(x,y) = max\{x+y-1, 0\}$, the *product conorm* $PC(x,y) = x+y-xy$ and the *Lukasiewicz conorm* $LC(x,y) = min\{x+y, 1\}$ $(x, y \in [0,1])$.

6.3. *Fuzzy comparisons of alternatives*

To conclude this Sec. 6, we shall briefly comment another classical fuzzy approach to aggregate preferences in a social choice context.

In this new approach, the individuals *compare the alternatives* that belong to a nonempty set or universe U.

In the crisp contexts, the usual way to compare alternatives is defining some ordering or binary relation $\mathcal{P}_i$ on U, understanding that $x\mathcal{P}_i y$ $(x, y \in U)$ means that the individual i prefers the alternative x to the alternative y. Observe that each relation P_i can be identified to a subset (also denoted P_i) of the product $U \times U$: $(x, y) \in P_i \iff x\mathcal{P}_i y$ $(x, y \in U)$.

In this *fuzzy* setting, however, the relations of preference that each individual defines *could be vague or imprecise*. In other words, instead of considering binary relations that are subsets of the cartesian product $U \times U$, we pass to consider *fuzzy relations* $\mathcal{A}_i$ that each individual i defines on U. Each fuzzy relation is a *fuzzy subset* of the cartesian product $U \times U$, given by a membership function $A_i : U \times U \to \mathcal{L}$. (Here $(\mathcal{L}, \preceq)$ is a suitable lattice with a greatest element $\bar{1}$ and a smallest element $\bar{0}$). In this context, given $N \geq 2, N \in \mathbb{N}$, a *social aggregation N-rule* would be a map $F_N : \mathcal{R}^N \to \mathcal{R}$, where $\mathcal{R}$ denotes the family of fuzzy relations that could be defined on U taking values on $\mathcal{L}$, or equivalently, the *fuzzy subsets* of $U \times U$ with corresponding lattice $(\mathcal{L}, \preceq)$. This is the approach considered in some recent papers.[49]

Thus, we should observe that *this approach is*, from a pure mathematical point of view, *analog to the former fuzzy approach* considered in this section, working now on $U \times U$ instead of directly on U. Indeed, from a purely abstract point of view, this approach is, now again, a particular case of the ones considered on the previous Secs. 3 and 4. The main set here is $\mathcal{R}$.

7. Further Comments: Other Functional Equations Arising in Social Choice Models

The key Theorem 4 characterizes associative bivariate means as semilatticial partial orders. From this key fact other functional equations appear in a natural way.

7.1. *Elections of best elements and the functional equations of consensus*

From the point of view of social choice, if F is an associative bivariate mean on a nonempty set X, given $x, y \in X$ the element $F(x, y) \in X$ can be understood as the final product of a *consensus* between x and y, that in a certain sense improves the quality of both elements $x, y \in X$. (Remember that $x \leq F(x, y)$ and $y \leq F(x, y)$ with respect to the semilatticial ordering "$\leq$" defined by means of F. Indeed $F(x, y) = x \vee y$).

Roughly speaking, we may interpret $F(x, y)$ as the *election of a best element* generated by x and y.

Since F is associative, this idea can be extended to any natural number $N \geq 2$. For instance, for $N = 3$ we may define a 3-rule F_3 as $F_3(x, y, z) = F(x, F(y, z))$ for every $x, y, z \in X$, and the element $F_3(x, y, z)$ can also be understood as the best element originated by x, y and z.

Having these ideas in mind, we may observe that if an agent changes her opinion and substitutes her choice x by the *consensus with the second agent* $F(x, y)$, whereas the second agent keeps her choice y, the final result of the election, namely

$F(F(x,y),y)$ should be the same best element. In other words, the functional equation $F(F(x,y),y) = F(x,F(x,y)) = F(x,y)$ must hold for every $x,y \in X$.

From an abstract point of view, we may introduce the following definition.

Definition 14. Let X be a nonempty set. Let F be a bivariate map $F : X^2 \to X$. The map F is said to satisfy the functional equation of *consensus* if $F(F(x,y),y) = F(x,F(x,y)) = F(x,y)$ for every $x,y \in X$.

(We point out that in this definition, F is not assumed to be an associative mean, but merely a bivariate map.)

Remark 8.

(i) As far as we know, the functional equation of consensus is new in the literature.

(ii) We immediately observe that if F is associative and satisfies the unanimity condition, then it also satisfies the functional equation of consensus. Actually $F(F(x,y),y) = F(x,F(y,y)) = F(x,y)$ and $F(x,F(x,y)) = F(F(x,x),y) = F(x,y)$ $(x,y \in X)$.

The converse is not true: On the one hand, a constant bivariate map obviously satisfies the functional equation of consensus, but it fails to satisfy the unanimity condition if X has at least two elements. On the other hand, there are examples of bivariate maps that satisfy the consensus equation as well as the unanimity condition, without being associative. Consider the following *example*: $X = \{a,b,c\}$. The bivariate map $F : X^2 \to X$ is given by $F(a,a) = F(a,b) = F(b,a) = a$; $F(b,b) = F(b,c) = F(c,b) = b$; $Fc,c) = F(a,c) = F(c,a) = c$. Notice that $F(a,F(b,c)) = F(a,b) = a$ but $F(F(a,b),c) = F(a,c) = c$.

The functional equation of consensus just considered appears as a natural functional equation coming from social choice contexts (having in mind the idea of best selections). It can be extended to more variables as next definition shows.

Definition 15. Let X be a nonempty set. Given a natural number $N \geq 2$, a map $F_N : X^N \to X$ is said to satisfy the *N-dimensional equation of consensus* if for every $i \in \{1\dots,N\}$ it holds that $F_N(x_1,\dots,x_i,\dots,x_N) = F_N(x_1,\dots,x_{i-1},F_N(x_1,\dots,x_N),x_{i+1},\dots,x_N)$ for every $(x_1,\dots,x_N) \in X^N$.

Despite associative bivariate means immediately generate N-means for any $N \geq 2$, we do not know if a similar result can be obtained for bivariate maps that satisfy the consensus equation. Actually, we do not know if starting from bivariate maps that satisfy the functional equation of consensus, we can generate for any natural number $N \geq 2$, maps on N variables that satisfy the N-dimensional equation of consensus. We leave it as an *open question*.

7.2. *Other functional equations*

Suppose now that a nonempty set X is endowed with a semilatticial partial order $\leq$. We already know that this generates an N-mean F_N for every natural number

$N \geq 2$, given by $F_N(x_1, \ldots, x_N) = x_1 \vee \ldots \vee x_N$ for every $(x_1, \ldots, x_N) \in X^N$. If we define the map $G : X^2 \to X$ by $G(x, y) = F_N(x \ldots (n-1 \text{ times}) \ldots x, y)$ $(x, y \in X)$ we obtain again that $G(x, y) = x \vee y$. Similarly $F_N(y \ldots (n-1 \text{ times}) \ldots y, x) = G(y, x) = G(x, y)$. In the same way, for every natural number k such that $1 \leq k \leq N$ we also have that $F_N(x, \ldots (k \text{ times}) \ldots x, y \ldots (n-k \text{ times}) \ldots y) = x \vee y$.

Fixed a natural number k such that $1 \leq k \leq N$ we consider the map $F_{N.k} : X^2 \to X$ given by $F_{N.k}(x, y) = F_N(x, \ldots (k \text{ times}) \ldots x, y \ldots (n-k \text{ times}) \ldots y)$ for every $x, y \in X$.

With this notation, we observe that:

$F_N(x, \ldots (k\text{times}) \ldots x, y \ldots (n-k \text{ times}) \ldots y) = F_{N.k}(x, y) = F_{N.k}(y, x) = F_{N.n-k}(x, y) = F_N(x, \ldots (n-k\text{times}) \ldots x, y \ldots (k \text{ times}) \ldots y)$ for every $x, y \in X$. This reminds us the functional equation of *migrativity*.[50] Fixed a number $t \in [0, 1]$, the functional equation of *t-migrativity*, typical in the study of aggregation functions in fuzzy contexts, is usually defined as $G(t \cdot x, y) = G(x, t \cdot y)$ for every $x, y \in [0, 1]$, where $G : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is the unknown function.

Needless to say that several other functional equations could possibly arise in these contexts of social choice, either in the crisp or fuzzy settings. Their study could constitute an interesting framework to be explored in next future.

8. Conclusions

Since the appearance of the Arrovian impossibility theorem in mathematical social choice, the ideas concerning the restrictions that we should impose to the aggregation rules, that are built to get a social choice preference from the individual profiles, must be relaxed and consequently changed, in order to construct new models where some possibility results could appear.

A common feature encountered in some of those new models is that the aggregation rules must satisfy a suitable system of functional equations.

In many important cases, among those equations the neutrality and unanimity are always present, and some other conditions related to extra (topological or algebraical) structure are involved.

Although the existence of aggregation rules in several key new models has already been characterized in the literature (e.g., in the topological or algebraical Chichilnisky models) it is important to study such functional equations by themselves.

Since those functional equations could always have solutions if no topological or algebraical restriction is added, we may conclude that *the failure of the existence of aggregation rules in some topological or algebraical models is indeed due to that (topological or algebraical) extra structure.*

The simultaneous solutions of the neutrality equation plus the unanimity equation are called general means, and it is proved that the existence of an associative bivariate mean induces the existence of general N-means for any $N \geq 2,\ N \in \mathbb{N}$. For this reason, we also pay attention to the study of the associativity equation.

As far as we know, this approach to the study of topological or algebraical Chichilnisky models through functional equations (introduced in the present paper) *is new in the social choice literature.*

The systematic use of functional equations to deal with a large variety of social choice aggregation models is, in our opinion, one of our main contributions.

The consideration of a fuzzy setting to aggregate preferences or choices into a collective one, being a classical item in the social choice literature, suggests us to analyze which functional equations are involved in some fuzzy approaches.

However, we prove that in some of the most classical fuzzy models introduced to aggregate preferences, the functional equations involved are again those that deal with general means, as well as the associativity equation, so that from a pure abstract point of view, in what concerns the searching for aggregation rules those fuzzy models are, roughly speaking, isomorphic to the crisp models previously analyzed.

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