The consensus functional equation in Agreement Theory

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Abstract We introduce the concept of the consensus functional equation, for a bivariate map defined on an abstract choice set. This equation is motivated by miscellaneous examples coming from different contexts. In particular, it appears in the analysis of sufficiently robust agreements arising in Social Choice. We study the solutions of this equation, relating them to the notion of a rationalizable agreement rule. Specific functional forms of the solutions of the consensus functional equation are also considered when the choice sets have particular common features. Some extension of the consensus equation to a multivariate context are also explored.

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1 Introduction

Assume that a research team, working individually, can reach a best output, say x. In the same way, a second (different) team, working individually, can reach their best performance, say y. However, if both teams collaborate and work together, they could reach an even better achievement, say $F(x,y)$. In this situation we may think that if one of the research teams could be able to get, working individually, the best

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possible output $F(x,y)$, the collaboration with the other team would not lead beyond that best attainment $F(x,y)$. In other words, the following *functional equation* arises:

$$F(x,y) = F(F(x,y),y) = F(x,F(x,y)).$$

Here $x,y \in X$, where X denotes he set of all possible goals of any research team.

This functional equation was already introduced in [11] and [13], where it was named the *consensus functional equation*.

The same equation arises when we study some particular kinds of agreements between two individuals, encountered in Social Choice. In this direction, let us assume that X is a nonempty set, that represents the collection of possible choices of each individual. (i.e.: X can be interpreted as the *choice set*, which is the same for both individuals). Suppose that $F : X \times X \to X$ is the map or that expresses the agreement between them. (i.e.: F can be interpreted as an *agreement rule*). In other words, if the first individual chooses the alternative $x \in X$ while the second individual chooses the alternative $y \in X$, then $F(x,y) \in X$ is another alternative, where the two agents agree, and so-to-say represents a *consensus option* for both agents.

The consensus equation is based upon the following idea: Suppose that we consider a situation in which the agreement is so *robust* that, if either of the individiuals changes her/his initial position on the one agreed by both of them, then the former achieved agreement should not vary, and remains the same. In formula, when analyzing this kind of agreements we should study the functional equation

$$F(F(x,y),y) = F(x,F(x,y)) = F(x,y), \ (x,y \in X).$$

In this second example, we may notice that the last equality of the formula is exactly the *unanimity principle* over the alternatives that are in the codomain of the map F (i.e., $F(z,z) = z$, for every $z \in F(X \times X) \subseteq X$). Thus, if F satisfies *non-imposition* condition (i.e., F is surjective, so that $X = F(X \times X)$), then the consensus equation implies the unanimity principle for the whole X.

The consensus equation transpires a property that, in some sense, reminds us the Nash equilibrium concept coming from Game Theory (see e.g. [26, 20]). As a matter of fact, if, for a given $x,y \in X$, we interpret $F(x,y)$ as the "best social agreement" (provided that the first agent chooses x whereas the second one chooses y), then the "best choice" for the first agent in order to reach that "best collective agreement", provided that the second agent keeps at her/his choice y, is to single out $F(x,y)$. The same argument applies for the second individual.

Here we furnish two further *examples*, that may also constitute a *motivation* to study the functional equation of consensus by its own merit:

1. *Suppose that several different tasks must be done to achieve a goal. Let T be the set of those tasks to be done. We may identify an individual with the subset of tasks that she/he is able to do. In this case, we may interpret X as the power set $\mathscr{P}(T)$, and consider the union "$\cup$" of subsets of T as a binary operation defined on $\mathscr{P}(T)$. Then it is clear that $(x \cup y) \cup y = x \cup (x \cup y) = x \cup y$ for every $x,y \in$*

$\mathscr{P}(T)$. Therefore, if we change the notation, setting $F(x,y) = x \cup y$ $(x,y \in X)$ we immediately get

$$F(F(x,y),y) = F(x,F(x,y)) = F(x,y) \quad (x,y \in X).$$

2. *Suppose that a nonempty set X is given a total order "$\preceq$". Given two elements $x,y \in X$, let $F(x,y) = y$ if $x \preceq y$, otherwise let $F(x,y) = x$ $(x,y \in X)$. It is obvious that this bivariate map $F : X \times X \to X$ satisfies, in particular, the functional equation*
$$F(F(x,y),y) = F(x,F(x,y)) = F(x,y) \quad (x,y \in X).$$

The paper is organized as follows:

Section 2 contains the basic background.

In Section 3 we study the consensus equation from an abstract point of view. To that end, we introduce a key concept; namely, that of a *rationalizable bivariate map*. Rationalizability is a notion that resembles the one already introduced in the literature of single-valued choice functions (see [4, 5, 28] or, more recently, [25]). Here, it means that a particular binary relation, that we call the *revealed relation*, describes F. (Thus, a map $F : X \times X$, on a choice set X, is said to be *rationalizable* if it can be expressed in terms of a suitable binary relation defined on X, as stated in Definition 5 in Section 2 below). We characterize bivariate maps that satisfy the consensus equation plus the *anonymity principle* as those that are rationalizable.

Associativity (i.e., $F(F(x,y),z) = F(x,F(y,z))$, for every $x,y,z \in X$) is a slightly more demanding property than the fulfilment of the consensus equation (see [11, 13]). When an agreement rule $F : X \times X \to X$ is associative, we get a more appealing result: namely, in this case there is a partial order, say $\preceq$, defined on X such that $(X,\preceq)$ is a *semi-lattice* and $F(x,y)$ turns out to be the supremum, with respect to $\preceq$, of $\{x,y\}$, for every $x,y \in X$ (see [11]). Notice that associativity can be viewed as an *extension* property (see e.g. [21, 16, 12, 22, 1, 29]). That is, if F is associative then, for any finite number of agents, we can induce agreement rules based on it. For instance, if the number of agents is three we may induce a trivariate rule $G : X \times X \times X \to X$ by declaring that $G(x,y,z) = F(F(x,y),z) = F(x,F(y,z))$ $(x,y,z \in X)$. In other words, associativity invites everyone "to join the party".

We also pay attention to the case where F is a *selector* (see [23, 18]), i.e., $F(x,y) \in \{x,y\}$, for every $x,y \in X$. In this case, and for obvious reasons, we will say that F satisfies the *independence of irrelevant alternatives condition*. Quite surprisingly, the independence of irrelevant alternatives condition is proved to be more restrictive than consensus.

In Section 4 we study several aspects of the solutions of the consensus equation in concrete scenarios.

In particular, we pay attention to the case in which X can be identified to a real interval. In this case, we add some extra conditions on F, namely monotonicity (Paretian properties) and continuity. Then we get some impossibility as well as some possibility results about the existence of agreement rules. On the one hand, we prove that there is no strongly Paretian bivariate map which satisfies consensus. On the other hand, we show that the only continuous agreement rules that satisfy the independence of irrelevant alternatives condition are the max and the min (i.e., those based upon the most and the least favoured individuals, respectively).

In Section 5 we explore the extended consensus equation, considering n-variate maps that correspond to general models of consensus where n-individuals are involved, obviously with $n \geq 2$.

A final Section 6 of further comments closes the paper.

Remark 1. Throughout the paper we will focus on the consensus equation involving only *two variables*. The generalization of this equation for more than two variables could constitute the raw material to build future pieces of research.

2 Preliminaries

In what follows, X will denote a nonempty set, that we interpret as the choice set (or the set of alternatives). Moreover, $F : X \times X \to X$ will be a bivariate map defined on X.

Definition 1. The map F is said to satisfy:

(1) the *unanimity principle* if $F(x,x) = x$ for every $x \in X$,
(2) the *anonymity principle* if $F(x,y) = F(y,x)$ for every $x,y \in X$,
(3) the *consensus functional equation (short, consensus)* if it holds that $F(F(x,y),y) = F(x,F(x,y)) = F(x,y)$ for every $x,y \in X$,
(4) the *associativity equation* if $F(x,F(y,z)) = F(F(x,y),z)$ for every $x,y,z \in X$,
(5) the *independence of irrelevant alternatives condition (shortly denoted by IIA))* if $F(x,y) \in \{x,y\}$, for every $x,y \in X$.

Remark 2. Needless to say that the word "*consensus*" is encountered in many branches of mathematical Social Choice theory, under a wide sort of scopes and approaches (see e.g. [15, 27, 30, 10, 14, 8, 19, 24, 6, 9, 20, 2]). All these contexts, as the equation introduced throughout the present manuscript, transpire the idea of buying models to interpret situations in which a "social agreement" between individuals is reached, following some "rules" or procedures.

Definition 2. A bivariate map $F : X \times X \to X$ is said to be an *agreement rule* if it satisfies the conditions (1) to (3) of Definition 1 above.

Now we recall some basic concepts on binary relations. A binary relation $\preceq$ defined on X is said to be a *partial order* if it is reflexive (i.e., $x \preceq x$ holds for every $x \in X$), antisymmetric (i.e., $(x \preceq y) \wedge (y \preceq x) \Rightarrow x = y$, for every $x,y \in X$) and transitive (i.e., $(x \preceq y) \wedge (y \preceq z) \Rightarrow x \preceq z$, for every $x,y,z \in X$). If, in addition, $\preceq$ is total (i.e., $(x \preceq y) \vee (y \preceq x)$ holds for every $x,y \in X$), then $\preceq$ is said to be a *total order*.

A binary relation $\mathscr{R}$ defined on X is said to have the *supremum property* if, for every $x, y \in X$, there is a unique $z \in X$ such that the following two conditions are met: (i) $(x\mathscr{R}z) \wedge (y\mathscr{R}z)$ holds; (ii) if there is $u \in X$ such that $x\mathscr{R}u$ and $y\mathscr{R}u$ hold, then $z\mathscr{R}u$ also holds. The unique element z that satisfies conditions (i) and (ii) is called the *supremum* of x and y and it is denoted by $sup_{\mathscr{R}}\{x,y\}$. Whenever $sup_{\mathscr{R}}\{x,y\} \in \{x,y\}$, then it is called *maximum* of x and y and it is denoted by $max_{\mathscr{R}}\{x,y\}$.

Definition 3. Let $\preceq$ be a partial order defined on X. Then $(X, \preceq)$ is said to be a *semi-lattice* if $\preceq$ has the supremum property.[1]

3 Consensus vs. rationalizable bivariate maps

The main purpose of this section is to provide a description of the agreement rules defined on a nonempty choice set X in terms of certain binary relations on X with special features. To that end, the following concept will play an important role.

Definition 4. Let X be a nonempty set. Let F be a bivariate map defined on X. Associated with F we consider on X a new binary relation, denoted by $\mathscr{R}_r$ and defined as follows: $x\mathscr{R}_r y \iff F(x,y) = y$, for every $x, y \in X$. The binary relation $\mathscr{R}_r$ is said to be the *revealed relation* of F.

Before introducing the notion of a rationalizable bivariate map, a notational convention is needed.

Notation. Let $\mathscr{R}$ be a binary relation defined on X. Then, for each $x \in X$, $G_{\mathscr{R}}(x)$ will denote the *upper contour set* of x, i.e., $G_{\mathscr{R}}(x) = \{z \in X : x\mathscr{R}z\}$.

Definition 5. A bivariate map F on X is said to be *rationalizable* if $F(x,y) \in G_{\mathscr{R}_r}(x) \cap G_{\mathscr{R}_r}(y)$, for every $x, y \in X$.

Remark 3. That is, the concept of rationalizability intends to describe a bivariate map by means of the upper contour sets of its corresponding revealed relation.

Next Theorem 1 characterizes the bivariate anonymous maps that satisfy the consensus equation in terms of those which are rationalizable.

Theorem 1. *Let F be a unanimous and anonymous bivariate map defined on X. Then F is rationalizable if and only if it satisfies consensus.*

Proof. Suppose that F is an anonymous bivariate map defined on X which satisfies the consensus equation. Let $x, y \in X$ be fixed. In order to show that F is rationalizable, notice that $F(x, F(x,y)) = F(x,y)$ since F satisfies the consensus

[1] For an excellent account of the material related to latticial or semi-latticial structures, see e.g., [7, 17].

equation. Thus, by definition of $\mathscr{R}_r$, $F(x,y) \in G_{\mathscr{R}_r}(x)$. Moreover, by anonymity together with consensus, it holds that $F(y,F(x,y)) = F(F(x,y),y) = F(x,y)$. Therefore, $F(x,y) \in G_{\mathscr{R}_r}(y)$. So, $F(x,y) \in G_{\mathscr{R}_r}(x) \cap G_{\mathscr{R}_r}(y)$. Since x,y are arbitrary elements of X, it follows that F is rationalizable.

For the converse, suppose that F is an anonymous rationalizable bivariate map defined on X. We want to see that F satisfies consensus. To that end, let $x,y \in X$ be fixed. Since F is rationalizable, it holds that $x\mathscr{R}_r F(x,y)$ and $y\mathscr{R}_r F(x,y)$. But, by definition of the revealed consensus relation, this means that $F(x,F(x,y)) = F(x,y)$ and $F(y,F(x,y)) = F(x,y)$. Now, by anonymity, $F(y,F(x,y)) = F(F(x,y),y)$ and therefore $F(F(x,y),y) = F(x,y)$. The fact that $F(F(x,y),F(x,y)) = F(x,y)$ follows directly from unanimity. Since x,y are arbitrary elements of X, we have shown that F satisfies consensus. $\square$

Remark 4. A unanimous and anonymous bivariate map F defined on X may fail to be rationalizable, even when $\mathscr{R}_r$ is transitive. Indeed, let $X = \{x,y,z\}$ and define $F : X \times X \to X$ as follows: $F(x,x) = F(y,z) = F(z,y) = x$, $F(y,y) = F(x,z) = F(z,x) = y$ and $F(z,z) = F(x,y) = F(y,x) = z$. It is clear that this map F is unanimous and anonymous. In addition, an easy calculation gives: $\mathscr{R}_r = \{(x,x),(y,y),(z,z)\}$. In other words, $x\mathscr{R}_r x$, $y\mathscr{R}_r y$ and $z\mathscr{R}_r z$ are the only possible relationships, according to $\mathscr{R}_r$, among the three elements of X. Thus, in addition to being reflexive and antisymmetric, $\mathscr{R}_r$ is transitive too. However, F is not rationalizable since, for example, $z = F(x,y) \notin G_{\mathscr{R}_r}(x) \cap G_{\mathscr{R}_r}(y)$.

In general, as the next Proposition 1 shows, for a (unanimous) bivariate map F, consensus is a less restrictive condition than associativity or independence of irrelevant alternatives.

Proposition 1. *Let F be a bivariate map defined on X.*

(i) If F is unanimous and associative, then it safisfies consensus.
(ii) If F satisfies IIA, then it safisfies consensus.

Proof. (i) Let $x,y \in X$ be fixed. Then, by associativity and unanimity, it holds that $F(F(x,y),y) = F(x,F(y,y)) = F(x,y)$. The other equality of consensus is proved similarly. So, since x,y are arbitrary points of X, F satisfies consensus.

(ii) Let $x,y \in X$ be fixed. Since F satisfies IIA, either $F(x,y) = x$ or $F(x,y) = y$. If $F(x,y) = x$, then we have that $F(F(x,y),y) = F(x,y) = x = F(x,x) = F(x,F(x,y))$. Now, if $F(x,y) = y$, then $F(F(x,y),y) = F(y,y) = y = F(x,y) = F(x,F(x,y))$. So, in any of the two cases, we have that $F(F(x,y),y) = F(x,F(y,y)) = F(x,y)$. Since x,y are arbitrary points of X, F satisfies consensus. $\square$

As a direct consequence of Theorem 1 and Proposition 1 we obtain the following corollary.

Corollary 1. *(i) Every unanimous, anonymous and associative bivariate map defined on X is rationalizable.*

(ii) Every bivariate map defined on X which satisfies IIA is rationalizable.

Remark 5. It is easy to see that, for a unanimous and anonymous bivariate map F, associativity and IIA are independent conditions. Moreover, there are agreement rules (hence rationalizable bivariate maps) other than associative maps or those that satisfy IIA. For a thorough description of the links that can be established among the mentioned properties of bivariate maps, see [13].

We now focus on associative agreement rules. As we have just seen, associativity is more restrictive than consensus. Indeed, associativity reinforces in a significant manner the scope of Theorem 1, as next Theorem 2 states.

Theorem 2. *Let F be an associative agreement rule defined on X. Then, $(X, \mathscr{R}_r)$ is a semi-lattice and $F(x,y) = sup_{\mathscr{R}_r}\{x,y\}$, for every $x, y \in X$.*

Proof. Let us first prove that $\mathscr{R}_r$ is a partial order on X. Indeed, reflexivity follows directly from unanimity of F. To see that $\mathscr{R}_r$ is antisymmetric, let $x, y \in X$ be such that $x\mathscr{R}_r y$ and $y\mathscr{R}_r x$ hold. Then, by definition of $\mathscr{R}_r$, we have that $F(x,y) = y$ and $F(y,x) = x$. So, by anonymity, $x = y$ and therefore $\mathscr{R}_r$ is antisymmetric. To prove transitivity of $\mathscr{R}_r$, let $x, y, z \in X$ be such that $x\mathscr{R}_r y$ and $y\mathscr{R}_r z$ hold. Then, by definition of $\mathscr{R}_r$ again, we have that $F(x,y) = y$ and $F(y,z) = z$. Let us see that $F(x,z) = z$, which would mean that $x\mathscr{R}_r z$. Indeed, $F(x,z) = F(x, F(y,z)) = F(F(x,y),z) = F(y,z) = z$, the second equality being true since F is associative. Thus, $\mathscr{R}_r$ is transitive too.

Let us show now that $(X, \mathscr{R}_r)$ is a semi-lattice. To that end, we must prove that, for given arbitrary elements $x, y \in X$, the supremum $sup_{\mathscr{R}_r}\{x,y\}$ exists. Notice that, since F is asociative, by Proposition 1 (i), it satisfies consensus too. So, $F(x, F(x,y)) = F(x,y)$ and therefore, by definition of $\mathscr{R}_r$, we have that $x\mathscr{R}_r F(x,y)$. In a similar way, now using anonymity and consensus, we get $F(y, F(x,y)) = F(F(x,y), y) = F(x,y)$. That is, $y\mathscr{R}_r F(x,y)$. So $F(x,y)$ is an upper bound, with respect to $\mathscr{R}_r$, of x and y. Let us see that it is the least upper bound. To see this, let $z \in X$ such that $x\mathscr{R}_r z$ and $y\mathscr{R}_r z$. Then, by definition of $\mathscr{R}_r$ again, we have that $F(x,z) = F(y,z) = z$. Hence $F(F(x,y),z)) = F(x, F(y,z)), y) = F(x,z) = z$, the first equality being true by associativity. Therefore, it follows that $F(F(x,y),z)) = z$ which means that $F(x,y)\mathscr{R}_r z$. So, we have shown that $F(x,y) = sup_{\mathscr{R}_r}\{x,y\}$, which proves the second claim of the statement of Theorem 2. This finishes the proof. $\square$

We now present some illuminating observations about the concepts introduced above.

Remark 6. (i) It should be observed that, if $\mathscr{R}$ is a binary relation on X for which $(X, \mathscr{R})$ is a semi-lattice, then the bivariate map $F_{\mathscr{R}}$ defined on X as $F_{\mathscr{R}}(x,y) = sup_{\mathscr{R}}\{x,y\} \in X \;\; (x, y \in X)$ is an associative agreement rule. Moreover, in this case, it can be easily proved that $\mathscr{R}$ and $\mathscr{R}_r$ coincide. So, associative agreement rules are characterized as those that can be rationalized by means of semi-latticial structures.

(ii) An agreement rule that satisfies IIA need not be associative. Moreover, and unlike the associative case, the revealed relation $\mathscr{R}_r$ in this situation can exhibit intransitivities. To see an example, consider the set $X = \{x, y, z\}$ and the bivariate map $F : X \times X \to X$ given by $F(x,x) = F(x,z) = F(z,x) = x$; $F(x,y) = F(y,x) =$

$F(y,y) = y$; $F(y,z) = F(z,y) = F(z,z) = z$. It is clear that F is anonymous and satisfies IIA. However, it is not associative since $F(x,F(y,z)) = F(x,z) = x$, whereas $F(F(x,y),z) = F(y,z) = y$. In terms of the revealed relation $\mathscr{R}_r$ we have that $x\mathscr{R}_r y$, $y\mathscr{R}_r z$ and $z\mathscr{R}_r x$. So, there is a "cycle", with respect to $\mathscr{R}_r$, for the three-element set $\{x,y,z\}$.

(iii) If an agreement rule F satisfies IIA, then the revealed consensus relation $\mathscr{R}_r$ becomes a total order on X. Moreover, if an agreement rule F which satisfies IIA is also associative, then $F(x,y) = max_{\mathscr{R}_r}\{x,y\}$, for every $x,y \in X$. So, associative agreement rules that satisfy IIA are characterized as those that can be rationalized by means of totally ordered structures.

(iv) Associative agreement rules have an interesting property that we call the *extension property*. The extension property means that an associative (bivariate) agreement rule generates associative, unanimous and anonymous n-variate rules, for any finite number of agents $n \in \mathbb{N}$. In other words, if a (unanimous and anonymous) map involving just two individuals is associative then it is possible that more and more individuals can "join the party" and enjoy a "stable" agreement. So, from a behavioural perspective, associativity is an appealing property. Indeed, let F_2 be an associative (bivariate) agreement rule. Then, by Theorem 2, $F_2(x,y) = sup_{\mathscr{R}_r}\{x,y\}$, for every $x,y \in X$. Now, for any $n \geq 3$, define $F_n : X^n = X \times \ldots$ (n-times) $\ldots \times X \to X$ as follows: $F_n(x_1,\ldots,x_n) = sup_{\mathscr{R}_r}\{x_1,\ldots,x_n\}$, for every $x_1,\ldots,x_n \in X$. It is then straightforward to see that, for every $n \geq 3$, the n-variate map F_n so-defined is associative, unanimous and anonymous.

(v) It should be noted that Theorem 2 can be applied to scenarios in which the choice set X is, on its own, a space of preferences. Indeed, let X denote the collection of all the total preorders (i.e.: transitive and total binary relations) that can be defined on a finite set Z. Let $F : X \times X \to X$ be the *Borda rule* (see [25]). Then, it is straightforward to see that F is an associative agreement rule. Thus, Theorem 2 states that the Borda rule is entirely described by the revealed relation on X. Actually, it is simple to prove that, in this case, $\mathscr{R}_r$ is given as follows: $\precsim_1 \mathscr{R}_r \precsim_2$ if and only if $\precsim_2 \subseteq \precsim_1$ and $\prec_1 \subset \prec_2$, $(\precsim_1, \precsim_2 \in X)$. Here, $\prec$ stands for the asymmetric part of $\precsim$ (i.e., $x \prec y$ if and only if $\neg(y \precsim x)$, for every $x,y \in X$).

As seen in the proof of Theorem 2, an associative, unanimous and anonymous bivariate map defined on X has the property that its revealed relation turns out to be transitive. The converse is not true even though the bivariate map is rationalizable (or, equivalently by Theorem 1, it satisfies consensus). Nevertheless, for a unanimous bivariate map that satisfies IIA, transitivity of its revealed relation implies associativity. These two facts are proved through the next Proposition 2, which closes this section.

Proposition 2. *(i) An agreement rule such that its associated revealed relation is transitive may fail to be associative.*
(ii) Every agreement rule that satisfies IIA is associative.

Proof. (i) Let $X = \{x,y,z,u\}$. Let $F : X \times X \to X$ be the bivariate map given by $F(x,x) = x$; $F(y,y) = y$; $F(x,y) = F(x,z) = F(y,z) = F(y,z) = F(z,x) = F(z,y) =$

$F(z,z) = F(z,u) = F(u,z) = z$; $F(x,u) = F(y,u) = F(u,x) = F(u,y) = F(u,u) = u$. It is clear that F satisfies unanimity and anonymity. Let us see that it is an agreement rule (i.e., it satisfies consensus) by showing that it is rationalizable (see Theorem 1 above). To that end, let $\mathscr{R}_r$ be its revealed relation. A direct calculation proves that $\mathscr{R}_r$ is given by: $x\mathscr{R}_r x$, $x\mathscr{R}_r z$, $x\mathscr{R}_r u$, $y\mathscr{R}_r y$; $y\mathscr{R}_r z$, $y\mathscr{R}_r u$, $z\mathscr{R}_r z$, $u\mathscr{R}_r z$, $u\mathscr{R}_r u$. Let us observe that $\mathscr{R}_r$ is transitive. Now, by checking the upper contour sets of $\mathscr{R}_r$, we obtain: $G_{\mathscr{R}_r}(x) = \{x,z,u\}$; $G_{\mathscr{R}_r}(y) = \{y,z,u\}$; $G_{\mathscr{R}_r}(z) = \{z\}$; $G_{\mathscr{R}_r}(u) = \{z,u\}$. Thus $F(x,x) = x \in G_{\mathscr{R}_r}(x)$; $F(x,y) = z \in G_{\mathscr{R}_r}(x) \cap G_{\mathscr{R}_r}(y)$; $F(x,z) = z \in G_{\mathscr{R}_r}(x) \cap G_{\mathscr{R}_r}(z)$; $F(x,u) = u \in G_{\mathscr{R}_r}(x) \cap G_{\mathscr{R}_r}(u)$; $F(y,y) = y \in G_{\mathscr{R}_r}(y)$; $F(y,z) = z \in G_{\mathscr{R}_r}(y) \cap G_{\mathscr{R}_r}(z)$; $F(y,u) = u \in G_{\mathscr{R}_r}(y) \cap G_{\mathscr{R}_r}(u)$; $F(z,z) = z \in G_{\mathscr{R}_r}(z)$; $F(z,u) = z \in G_{\mathscr{R}_r}(z) \cap G_{\mathscr{R}_r}(u)$. Therefore F is rationalizable. Finally, observe that F is not associative since $F(F(x,y),u) = F(z,u) = z \neq u = F(x,u) = F(x,F(y,u))$.

(ii) Let $x,y,z \in X$ be fixed. We must show that $F(F(x,y),z) = F(x,F(y,z))$. Since F satisfies IIA, it follows that $F(x,y) \in \{x,y\}$, $F(x,z) \in \{x,z\}$ and $F(y,z) \in \{y,z\}$. So we distinguish among eight possibilities:

(1) $F(x,y) = x$, $F(x,z) = x$ and $F(y,z) = y$. In this case, $F(F(x,y),z) = F(x,z) = x = F(x,y) = F(x,F(y,z))$ and we are done.

(2) $F(x,y) = x$, $F(x,z) = x$ and $F(y,z) = z$. In this case, $F(F(x,y),z) = F(x,z) = x = F(x,z) = F(x,F(y,z))$ and we are done again.

(3) $F(x,y) = x$, $F(x,z) = z$ and $F(y,z) = y$. Now, since F is anonymous, $F(y,x) = F(x,y) = x$ and $F(z,y) = F(y,z) = y$. So we get $y\mathscr{R}_r x$ and $x\mathscr{R}_r z$. Thus, by transitivity of $\mathscr{R}_r$, it follows that $y\mathscr{R}_r z$. But $F(z,y) = F(y,z) = y$ means that $z\mathscr{R}_r y$ too. In addition, $\mathscr{R}_r$ is antisymmetric since F is anonymous. Therefore, $y = z$. Now, if $y = z$, $F(F(x,y),z) = F(x,F(y,z))$ becomes $F(F(x,y),y) = F(x,F(y,y))$ or, equivalently, $F(F(x,y),y) = x = F(x,y) = F(x,F(y,y))$ and we are done.

(4) $F(x,y) = x$, $F(x,z) = z$ and $F(y,z) = z$. In this case, $F(F(x,y),z) = F(x,z) = z = F(x,F(y,z))$ and we are done.

(5) $F(x,y) = y$, $F(x,z) = x$ and $F(y,z) = y$. In this case, $F(F(x,y),z) = F(y,z) = y = F(x,y) = F(x,F(y,z))$ and we are done.

(6) $F(x,y) = y$, $F(x,z) = x$ and $F(y,z) = z$. In this case, and arguing in the same way as in case (3) above, we have that $x\mathscr{R}_r y$ and $z\mathscr{R}_r x$ which, by transitivity, implies that $z\mathscr{R}_r y$. This, together with $y\mathscr{R}_r z$, implies that $y = z$. Then, $F(F(x,y),z) = F(x,F(y,z))$ becomes $F(F(x,y),y) = F(x,F(y,y))$ or, equivalently, $F(F(x,y),y) = y = F(x,y) = F(x,F(y,y))$ and we are done again.

(7) $F(x,y) = y$, $F(x,z) = z$ and $F(y,z) = y$. In this case, $F(F(x,y),z) = F(y,z) = y = F(x,y) = F(x,F(y,z))$ and we are done. Finally,

(8) $F(x,y) = y$, $F(x,z) = z$ and $F(y,z) = z$. In this case, $F(F(x,y),z) = F(y,z) = z = F(x,z) = F(x,F(y,z))$ which concludes the proof. $\square$

Remark 7. It can be shown that if X is a three-elements set (i.e, $X = \{x,y,z\}$), then any agreement rule defined on X for which $\mathscr{R}_r$ is transitive is, in fact, associative.

4 Possibility vs. impossibility results on the existence of agreement rules in continuum spaces

In this section, we study the consensus equation in particular contexts. In general, the solutions of this equation cannot be described in an easy way (see [13] for details). However, in some special cases, and imposing also some natural extra conditions on the map F, it is indeed possible to entirely describe its solutions. (See e.g. [23, 13] for further results in this direction).

Throughout this section, we assume that the choice set X is a real interval.

Both impossibility as well as possibility results arise. On the one hand, we prove that there is no strongly Paretian bivariate map which satisfies consensus. On the other hand, we show that the only continuous agreement rules that satisfy IIA are the max and the min.

In what follows, $\mathscr{I}$ will represent an interval of the real line $\mathbb{R}$.

Remark 8. At this stage, we point out that real intervals naturally arise to represent the set of alternatives in several contexts of Social Choice. Thus, X could be a set of monetary payoffs or, in a probabilistic scenario, X could represent the space of lotteries between two outcomes. In the first situation X can be identified as the real interval $[0,\infty)$. And in the second case, X can be identified as $[0,1]$.

Notation. Let $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ be a bivariate map. For every $x \in \mathscr{I}$, F_x (respectively, F^x) stands for the *vertical* (respectively, *horizontal*) *restriction* of F, that is, $F_x(y) = F(x,y) \in \mathscr{I}$ (respectively, $F^x(y) = F(y,x) \in \mathscr{I}$).

We recall the concept of an idempotent function defined on $\mathscr{I}$. This concept will play a significant role in the sequel, in particular in the next Proposition 3.

Definition 6. A function $f : \mathscr{I} \to \mathscr{I}$ is said to be *idempotent* if $f(f(x)) = f(x)$, for every $x \in \mathscr{I}$.

Proposition 3. *Let $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ be a bivariate map.*

(i) F is unanimous if and only if $F_x(x) = F^x(x) = x$ for every $x \in \mathscr{I}$.
(ii) F is anonymous if and only if $F_x(y) = F^x(y)$ for every $x,y \in \mathscr{I}$.
(iii) F satisfies consensus if and only if, for every $x \in X$, both restrictions F_x and F^x are idempotent functions and, for each $z \in F(\mathscr{I} \times \mathscr{I})$, it holds that $F_z(z) = F^z(z) = z$.

Proof. Parts (i) and (ii) follow directly. So we prove only part (iii). Suppose that F satisfies consensus and let $x \in X$ be fixed. Then, we have that $F_x(F_x(y)) = F(x,F_x(y)) = F(x,F(x,y)) = F(x,y) = F_x(y)$ for every $y \in \mathscr{I}$. Also, we have that $F^x(F^x(y)) = F(F^x(y),x) = F(F(y,x),x) = F(y,x) = F^x(y)$, for every $y \in \mathscr{I}$. Therefore, F_x and F^x are both idempotent functions. Since x is an arbitrary element of $\mathscr{I}$, we have proved that F_x and F^x are both idempotent functions for every $x \in \mathscr{I}$. The fact that, for each $z \in F(\mathscr{I} \times \mathscr{I})$, $F_z(z) = F^z(z) = z$ follows directly from consensus.

Conversely, suppose that, for every $x \in \mathscr{I}$, F_x and F^x are both idempotent functions. Let $x, y \in \mathscr{I}$ be fixed. Then, we have that $F(x, F(x,y)) = F_x(F_x(y)) = F_x(y) = F(x,y)$, and also we have that $F(x,y) = F^y(x) = F^y(F^y(x)) = F(F^y(x), y) = F(F(x,y), y)$. Moreover, $F(F(x,y), F(x,y)) = F(x,y)$ since, by hypothesis, $F_z(z) = F^z(z) = z$, for every $z \in F(\mathscr{I} \times \mathscr{I})$. Therefore, F satisfies consensus.

Remark 9. It should be noted that the concepts introduced above can indeed be defined in a more abstract setting. As a matter of fact, Proposition 3 remains true if $\mathscr{I}$ is replaced by a nonempty choice set X.

Before presenting a basic definition of the most familiar notions involving monotonicity properties of real-valued bivariate functions, we recall that given $(x,y), (u,v) \in \mathscr{I} \times \mathscr{I}$, the notation $(x,y) \le (u,v)$ means that both $x \le u$ and $y \le v$ hold. Similarly, $(x,y) < (u,v)$ means that both $(x,y) \le (u,v)$ and $(x,y) \neq (u,v)$ hold. Finally, $(x,y) \ll (u,v)$ means that both $x < u$ and $y < v$ hold.

Definition 7. A bivariate map $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ is said to be:

(i) *Paretian (or non-decreasing)* if $(x,y) \le (u,v)$ implies $F(x,y) \le F(u,v)$, for every $x,y,u,v \in \mathscr{I}$.
(ii) *weakly Paretian* if $(x,y) \ll (u,v)$ implies $F(x,y) < F(u,v)$, for every $x,y,u,v \in \mathscr{I}$.
(iii) *strongly Paretian* if $(x,y) < (u,v)$ implies $F(x,y) < F(u,v)$, for every $x,y,u,v \in \mathscr{I}$.
(iv) *dictatorial* if either $F(x,y) = x$ for every $x,y \in \mathscr{I}$ holds, or $F(x,y) = y$ for every $x,y \in \mathscr{I}$ holds.
(v) *dichotomic* if for every $x \in \mathscr{I}$ the functions F_x and F^x are either constant or strictly increasing.
(vi) *continuous* if the inverse image of every Euclidean open subset of $\mathscr{I}$ is an open subset of $\mathscr{I} \times \mathscr{I}$, where $\mathscr{I} \times \mathscr{I}$ is endowed with the usual product (Euclidean) topology.

Remark 10. Notice that the monotonicity properties that appear in Definition 7 above are meaningful in the case that the choice set X is a set of monetary payoffs.

Now we present a general theorem that allows us to derive certain impossibility results. Before a simple and useful lemma concerning strictly increasing real-valued idempotent functions is shown.

Lemma 1. *Let $f : \mathscr{I} \to \mathscr{I}$ be a strictly increasing idempotent function. Then, $f(x) = x$, for every $x \in \mathscr{I}$. (In other words, the identity function is the only one strictly increasing real-valued function that is idempotent.)*

Proof. Let $x \in \mathscr{I}$ arbitrarily be given. Let us see that $f(x) = x$. If, on the contrary, $f(x) \neq x$ then either $f(x) < x$ or $x < f(x)$. Assume that $f(x) \neq x$. Then, since f is strictly increasing, we have that $f(f(x)) < f(x)$. But, since f is idempotent, $f(f(x)) = f(x)$. So, we get $f(x) < f(x)$, which is a contradiction. The case $x < f(x)$ is handled in a similar way.

Theorem 3. *Let $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ be a bivariate map. Then the following assertions are equivalent:*

i) F is dichotomic, unanimous and satisfies consensus.
ii) F is dictatorial.

Proof. (ii) implies (i) is routine. So, we will concentrate on (i) implies (ii). Assume then that F is dichotomic, unanimous and satisfies consensus. Let us prove first that there cannot exist $x \in \mathscr{I}$ such that both F_x and F^x are constant functions. Indeed, suppose, by way of contradiction, that there is $x_0 \in \mathscr{I}$ for which both F_{x_0} and F^{x_0} are constant functions. Then, since F is unanimous and therefore $F_{x_0}(x_0) = x_0$, it holds that $F_{x_0}(y) = x_0 = F^{x_0}(y)$, for all $y \in \mathscr{I}$. Now, let $x_1 \in \mathscr{I}$ so that $x_0 < x_1$ (if any). The case $x_1 < x_0$ (if any) is similar. Then, $F_{x_1}(x_0) = F^{x_0}(x_1) = x_0 < x_1 = F_{x_1}(x_1)$. So, since, by hypothesis, F is dichotomic, it follows that F_{x_1} is a strictly increasing function. Now, since F satisfies consensus, by Proposition 3(iii), F_{x_1} is idempotent. Thus, by the previous lemma, F_{x_1} is the identity map (i.e., $F_{x_1}(y) = y$, for all $y \in \mathscr{I}$). In a similar way, we can prove that F^{x_1} is the identity map. Therefore, we have shown, in fact, that, for every $z \in \mathscr{I}$ such that $x_0 < z$ (if any), both functions F_z as well as F^z are the identity map. Let now consider three points $x_0, x_1, x_2 \in \mathscr{I}$ such that $x_0 < x_1 < x_2$ (if any). Then, since F_{x_1} is the identity map, it follows that $F(x_2, x_1) = F_{x_2}(x_1) = x_1$. Now, by definition, we have that $F(x_2, x_1) = F^{x_1}(x_2)$. So, $F^{x_1}(x_2) = x_1 \neq x_2$, which contradicts the fact, shown above, that F^{x_1} is the identity function. Therefore, there cannot exist $x \in \mathscr{I}$ such that both F_x and F^x are constant functions.

Using a similar argument to that employed above we can prove that there cannot exist $x \in \mathscr{I}$ such that both F_x and F^x are strictly increasing functions. (The proof of this assertion is left to the reader).

So, we have proved that, for every $x \in \mathscr{I}$, if F_x is a constant (respectively, the identity) function, then F^x is the identity (respectively, a constant) function. Suppose now that, for some $x_0 \in \mathscr{I}$, F_{x_0} is constant and F^{x_0} is the identity. Let us show that this situation leads to the conclusion $F(x,y) = x$, for all $x, y \in \mathscr{I}$ (in other words, F is dictatorial, the first individual acting as a dictator). Indeed, let $x_1 \in \mathscr{I}$ so that $x_0 < x_1$ (if any). Then $F_{x_1}(x_0) = F(x_1, x_0) = F^{x_0}(x_1) = x_1$, the last equality being true since F^{x_0} is the identity. Now, by unanimity, $F_{x_1}(x_1) = x_1$. So, $F_{x_1}(x_0) = x_1 = F_{x_1}(x_1)$, hence, since F is dichotomic, it follows that $F_{x_1}(y) = x_1$, for all $y \in \mathscr{I}$. The case $x_1 < x_0$ (if any) is similar leading to the same conclusion (i.e., $F_{x_1}(y) = x_1$, for all $y \in \mathscr{I}$). Thus, $F(x,y) = F_x(y) = x$, for all $x, y \in \mathscr{I}$.

Suppose now that, for some $x_0 \in \mathscr{I}$, F_{x_0} is the identity and F^{x_0} is constant. Arguing in a similar manner as above, it can be seen now that $F(x,y) = F_x(y) = y$, for all $x, y \in \mathscr{I}$. This ends the proof.

Theorem 3 immediately gives rise to the following corollaries.

Corollary 2. *There is no dichotomic bivariate map $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ that satisfies unanimity, anonymity and consensus.*

Proof. Just observe that dictatorial bivariate maps on $\mathscr{I}$ are not anonymous.

Corollary 3. *There is no strongly Paretian bivariate map $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ that satisfies unanimity and consensus.*

Proof. It is also a straightforward consequence of Theorem 3. Indeed, suppose that there is a bivariate map, say F, that is strongly Paretian, unanimous and satisfies consensus. Then, since, clearly, strongly Paretian implies dichotomic, it follows, by Theorem 3, that F is dictatorial. But neither of the two dictatorial bivarite maps are strongly Paretian. This contradiction provides the result.

Remark 11. (i) A careful glance at the proof of Theorem 3 above shows that the only bivariate map on $\mathscr{I}$ which satisfies consensus and has the additional property that all of its vertical restrictions are strictly increasing functions (respectively, all of its horizontal restrictions are strictly increasing functions) is dictatorial over the second (respectively, first) coordinate. That is, $F(x,y) = y$ for every $x,y \in \mathscr{I}$ (respectively, $F(x,y) = x$ for every $x,y \in \mathscr{I}$).
(ii) If strongly Paretian is relaxed to Paretian (or weakly Paretian) then the impossibility result does not hold true. For example, consider the dictatorial bivariate maps or the max/min functions.

It is interesting to search for some possibilities results based on certain natural properties, in addition to consensus, of the bivarite map. In [13], it was offered a characterization of the maximum rule (i.e., $F(x,y) = \max\{x,y\}$), for the case $\mathscr{I} = \mathbb{R}$, in terms of five properties; namely, continuity, unanimity, anonimity, consensus and upper-Pareto. A bivarite map $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be *upper-Paretian* if it is non-decreasing and for every $x,y \in \mathbb{R}$ there exists $u \in \mathbb{R}$ such that $y < u$ and $F(x,y) < F(x,u)$. It is not difficult to show that this latter characterization result remains true if $\mathbb{R}$ is replaced by a real interval $\mathscr{I}$. A much more easy result can be obtained if a more demanding property than consensus is required; namely the fulfilment of IIA. We now state this possibility result. Actually, we establish that the only continuous bivariate maps $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ that satisfy IIA are the max, the min and the dictatorial functions. In particular, we have that the only continuous agreement rules that satisfy IIA are the max and the min functions.

Theorem 4. *Let $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ be a bivariate map. Then the following conditions are equivalent:*

(i) F is continuous and satisfies IIA.
(ii) F is of one of the following forms:
(1) $F(x,y) = x$, for every $x,y \in \mathscr{I}$.
(2) $F(x,y) = y$, for every $x,y \in \mathscr{I}$.
(3) $F(x,y) = max\{x,y\}$, for every $x,y \in \mathscr{I}$.
(4) $F(x,y) = min\{x,y\}$, for every $x,y \in \mathscr{I}$.

Proof. It is straightforward to see that (ii) implies (i).

To prove the converse implication, (i) implies (ii), let $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ be a continuous bivariate map which satisfies IIA. Let $x \in \mathscr{I}$ be fixed and consider the

vertical restriction F_x. Since F satisfies IIA, $F_x(y) \in \{x,y\}$ for all $y \in \mathscr{I}$. The continuity of F_x, together with IIA, clearly implies that F_x must be of one of the following types:
(1) $F_x(y) = x$, for every $y \in \mathscr{I}$.
(2) $F_x(y) = y$, for every $y \in \mathscr{I}$.
(3) $F_x(y) = y$, if $y \geq x$ and $F_x(y) = x$, if $y < x$.
(4) $F_x(y) = x$, if $y \geq x$ and $F_x(y) = y$, if $y < x$.

Now, the continuity of F (in two variables) clearly implies that if for some $x_0 \in \mathscr{I}$, F_{x_0} is of the type (i), i= 1 to 4, then F_x is of the type (i), for all $x \in \mathscr{I}$. Finally, it is straightforward to see that the situation for each of the four cases leads to the corresponding functional form given in the statement of the theorem.

As a direct consequence of Theorem 4 we obtain the following corollary.

Corollary 4. *Let $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ be a bivariate map. Then the following conditions are equivalent:*

(i) F is continuous, anonymous and satisfies IIA.
(ii) Either $F(x,y) = max\{x,y\}$ (for all $x,y \in \mathscr{I}$), or $F(x,y) = min\{x,y\}$ (for all $x,y \in \mathscr{I}$).

Remark 12. Theorem 4 strongly depends on the independence of irrelevant alternatives (IIA) condition imposed to F, since there are continuous bivariate maps $F : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ which satisfy consensus other than those belonging to the four types that appear in the statement of the theorem (for details, see [13]).

5 Extending the consensus equation to a multivariate context

Until now we have studied the consensus equation that involves only two factors in its definition. We now explore the extended consensus equation which means that we are going to consider the n-factors (or n-individuals) case. To that end, the next notation and definition are in order.

Notation. Let X be a set and let $n \in \mathbb{N}$. Let us denote by X^n the n-fold Cartesian product of X and consider a n-variate map $F : X^n \to X$. Let $\mathbf{x}= (x_j)_{j \in N} \in X^n$. In order to make the notation as simple as possible, let us denote by $\mathbf{x}^F \in X^n$ any of the 2^n elements of X^n derived from $\mathbf{x}$ in the following way: For every $j \in N$, $\mathbf{x}^F_j = x_j$, or $F(\mathbf{x})$.

Definition 8. A n-variate map $F : X^n \to X$ is said to satisfy the *extended consensus equation* if $F(\mathbf{x}^F) = F(\mathbf{x})$, for every $\mathbf{x} \in X^n$.

The following result states that if a n-variate unanimous map satisfies extended consensus then it can be fully described by a family of $(n-1)$-variate maps that satisfy extended consensus too, together with a kind of (weak) unanimity. So, the entire description of the class of bivariate maps that satisfy consensus is important since it allows us to also describe those that satisfy extended consensus for any number of factors (agents). Before presenting the result let us introduce the following notation.

Notation. Let $\mathbf{x}\in X^n$, $j \in N$ and $z \in X$ be given. Then by $\mathbf{x}_{+j}(z)$ we mean the following element of X^{n+1}: $\mathbf{x}_{+j}(z) = (\mathbf{x}_{+j}(z))_k = x_k$, if $k < j$, or z, if $k = j$, or x_{k-1}, if $k > j$. In words, the $j-1$ first components of $\mathbf{x}_{+j}(z)$ are the same as those of $\mathbf{x}$, the j-th component is z and the remaining components are those of $\mathbf{x}$ shifted one place on the right. Let now $\mathbf{x}\in X^n$ and $j \in N$ be given. Then $\mathbf{x}_{+j}$ will denote the element of X^{n-1} obtained by removing from $\mathbf{x}$ the j-th component while keeping the remaining components equal to those of $\mathbf{x}$.

In addition, 1_n will denote the vector of $\mathbb{R}^n$ with all the coordinates equal to one. Similarly, for any $x \in X$ given, $x1_n$ will stand for the element of X^n with all the components equal to x. Let $F : X^n \to X$ be a n-variate map. For every $x \in X$ and $j \in N$, denote by F_x^j the $(n-1)$-variate map defined as follows: $F_x^j(\mathbf{z}) = \mathbf{F}(\mathbf{z}_{+j}(x))$, for every $\mathbf{z}\in X^{n-1}$.

Once the above tedious notation has been introduced we are ready to offer the main result of this section.

Theorem 5. *A n-variate unanimous map $F : X^n \to X$ satisfies the extended consensus equation if and only if F_x^j does (for every $x \in X$, $j \in N$).*

Proof. Suppose first that F fulfils consensus. Let $x \in X$ and $j \in N$ be given and consider the $(n-1)$-variate map F_x^j. Assume, without loss of generality that $j = 1$. Then, for every $\mathbf{z}\in X^{n-1}$, it follows that $F_x^j(\mathbf{z}^{F_x^j}) = F(\mathbf{z}_{+j}^{F_x^j}(x)) = F(\mathbf{z}_{+j}(x)) = F_x^j(\mathbf{z})$, since F satisfies consensus. So, F_x^j fulfils consensus.

Conversely, assume now that, for each $x \in X$ and each $j \in N$, F_x^j satisfies consensus and let us prove that so F does. To that end, let $\mathbf{x}\in X^n$ be fixed and consider any of the 2^n elements $\mathbf{x}^F \in X^n$, as defined above. We distinguish between the two following cases: (i) There is at least one component of $\mathbf{x}^F$, say $\mathbf{x}_j^F \in N$, which is different from $F(\mathbf{x})$, or (ii) All the components of $\mathbf{x}^F$ are $F(\mathbf{x})$. If (i) occurs, then $F(\mathbf{x}^F) = F_{x_j}^j(\mathbf{x}_{-j}^F)$. Now observe that, for each $k \in N \setminus \{1\}$, the k-th component of $\mathbf{x}_{-j}^F$ is equal to x_k or equal to $F(\mathbf{x})$. So, since $F(\mathbf{x}) = F_{x_j}^j(\mathbf{x}_{-j})$ and, by hypothesis, $F_{x_j}^j$ satisfies consensus, it turns out that $F(\mathbf{x}^F) = F_{x_j}^j(\mathbf{x}_{-j}^F)^{F_{x_j}^j} = F(\mathbf{x})$. If (ii) happens, then the fact that $F(F(\mathbf{x}1_n)) = F(\mathbf{x})$ follows from the unanimity of F. So, the proof is ended.

Remark 13. It is interesting to study the functional form the of the unanimous n-variate maps that, in addition to fulfil the consensus equation, also satisfy natural conditions like anonymity or continuity. For the particular case $X = \mathbb{R}$, the class of unanimous n-variate maps that fulfil consensus plus continuity is closely related to the class of lattice polynomial functions (see [23] for a thorough discussion of these functions). Indeed, it is not difficult to see that a lattice polynomial function in $\mathbb{R}^n$ satisfies consensus, unanimity and continuity. Nevertheless, the class of real-valued functions defined on $\mathbb{R}^n$ that satisfy consensus, unanimity and continuity is larger than the class of lattice polynomial functions as the next example shows. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function given by: $F(x,y) = x$, if $y \leq 1$ and $x \geq y$, or $F(x,y) = x$, if

$y \le 1$ and $x \le y$, or $F(x,y) = x$, if $y \ge 1$ and $x \ge 1$, or $F(x,y) = 1$, if $y \ge 1$ and $x \le 1$. Then it is straightforward to see that F so-defined satisfies consensus, unanimity and continuity. Actually, it can be shown that $F(x,y) = \max\{x, \min\{y,1\}\}$.

If anonymity is added to the previous discussion then the class of lattice polynomial functions reduces to the so-called order statistics functions (for a discussion of this latter family, see also [23]). We conjecture that the class of real-valued functions defined on $\mathbb{R}^n$ that satisfy consensus, unanimity, anonymity and continuity agrees with the family of order statistics functions.

6 Further comments

One of the achievements in [13] is showing that under unanimity plus anonymity, a new functional equation for bivariate maps (namely, the so-called equation of consensus, also analyzed in the present manuscript), is indeed equivalent to a weaker version of the associativity equation.

Throughout the present paper, we have not intended to solve the functional equation of consensus in the general case of bivariate maps F defined on a nonempty set X. Indeed, we may observe that the even more restrictive condition of associativity leads to a too wide set of possible solutions. In this direction, a glance at [3] may give us an idea of how large could be the set of solutions, even in relevant particular cases (e.g. : $X = \mathbb{R}$ or $X = [0,1]$).

In what concerns the consensus equation, it is important to point out that, under unanimity plus anonymity, any finite sequence of applications of F in which only the elements $x, y \in X$ are involved[2] always leads to $F(x,y)$. Viewing $F(x,y)$ as an agreement rule defined by means of a binary operation $*_F$ on X (i.e. $F(x,y) = x *_F y$, for every $x, y \in X$), the algebraic structure $(X, *_F)$ could be understood as being a weakening of the notion of a semigroup, that is called a magma in the specialized literature. This magma, namely the set X matched with the operation $*_F$, has the aforementioned property of simplification for finite sequences. (See [13] for further details).

In particular cases the operation $*_F$ is semi-latticial. In other special cases, it is a selector. And in some more restrictive cases, it corresponds to the idea of taking a maximum as proved in Corollary 4 above. Obviously, this fact of "taking a maximum" strongly agrees with the underlying idea of "reaching the best possible agreement" or "selecting the best possible option" commonly encountered in any process of aggregation of individual alternatives into a social one, typical of a wide variety of Social Choice contexts.

[2] An example could be $(((y *_F ((x *_F y) *_F x)) *_F (y *_F x)) *_F (y *_F y)) *_F x$.

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