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Numerical Representability of Ordered Topological Spaces with Compatible Algebraic Structure

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Abstract We analyze the numerical representability of total preorders defined on semitopological real algebras through continuous order-preserving real-valued functions that are also additive and multiplicative. The results obtained are used to interpret important concepts arising in Social Choice theory.

Keywords Totally preordered topological spaces · Continuous numerical representations of total preorders · Totally preordered algebraic structures · Semitopological real algebras · Social Choice theory

Mathematics Subject Classifications (2010) Primary 54F05; Secondary 06F25

1 Introduction

The aim of the present note is to study, on totally preordered topological vector spaces, the existence of continuous numerical representations through real-valued order-preserving functions that are, in addition, algebraic homomorphisms.

This framework extends in a natural way classical results [32] as well as more recent contributions [8, 23, 26, 37] arising in the literature concerning continuous numerical representations of totally preordered topological spaces endowed with some additional compatible algebraic structure.

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In the classical works concerning ordered structures, one can find literature on orderings compatible with algebraic structures (see e.g. [6, 29]). Several of the classical sources include also some results concerning the existence of real-valued order-preserving numerical representations, but not paying attention to continuity. As a matter of fact, the papers devoted to *topological* spaces endowed with continuous orderings and compatible algebraic structure are scarce and they are mainly related to some kind of applications (e.g., in expected utility theory in Economics or Decision-Making, see [4, 28]). It is also noticeable that algebraic techniques have also recently been introduced in the search for classical numerical representations of totally preordered structures, as well as some other kinds of orderings, in a unified way (see e.g. [10, 11]).

A different but complementary set of results that share the idea of dealing with topological spaces endowed with a continuous ordering and a compatible algebraic structure relates to ordered topological *groups* and *semigroups* (see e.g. [16, 17, 22, 38]). Indeed, in several of these situations there are theorems of *automatic continuity*: to put an example, any translation-invariant linear order defined on a group is continuous as regards the order topology, and a group endowed with a translation-invariant total (linear) order is actually a topological group with respect to the order topology (see [38] for further details).

Another kind of algebraic structures on which the study of continuous numerical representability of total preorders through real-valued order-preserving functions that are also algebraic homomorphisms is the category of *totally preordered topological vector spaces*. Apart from the classical studies devoted to applications into Economics (e.g. in expected utility theory) some theoretical papers have also appeared in this literature (see e.g [21]). As a matter of fact, the problem concerning the existence of a continuous representation of a preordered topological vector space was already analyzed in the literature in the case that *the preorder is not necessarily total* (see [12]).

In the present work the main novelty with respect to the most recent literature is the analysis of numerical representations that are continuous, linear and *multiplicative*. This implies that the totally preordered topological vector space is endowed with an additional binary relation, say $*$, that is also compatible with the given preordering. Therefore our framework throughout the paper is the category of totally preordered semitopological real *algebras*.

An underlying motivation to study these kind of structures comes from applications in Economics and Social Sciences: Indeed, *linear* (or at least *additive*) real-valued order-preserving functions defined on totally preordered topological vector spaces arise in a natural way when considering models involving risk situations (expected or subjective utility theory) as mentioned above.

However, as a counterpart to linear representability, there are some other situations (arising mainly in $\mathbb{R}^n$ but extendable to a wider family of structures) where a new binary operation plays a role, and consequently *multiplicative* real-valued order-preserving maps are necessary. This class of real-valued order-preserving functions can be often encountered in Measurement theory, Psychology and Economics (see, e.g., [1, 20, 35]). In contexts coming from Economics, a typical example is the set of *Cobb-Douglas* functions defined on $\mathbb{R}^n$ (see e.g., [20], p. 156), that are multiplicative (but unfortunately, they may fail to be additive). Another typical situation appears when preferences over elements of $\mathbb{R}^n$ are involved in a way in which an element $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is understood as a *commodity bundle* or a *"basket of goods"*, so that if the basket $(x_1, x_2, \ldots, x_n)$ is preferred to the basket $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then for every coordinatewise change of scale the preference is kept: that is, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are strictly positive real numbers, the basket $(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$ should be preferred to the basket $(\alpha_1 y_1, \alpha_2 y_2, \ldots, \alpha_n y_n)$. From a purely abstract point of view, the basket $(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$ appears as a *coordinatewise product* (say $*$) defined on $\mathbb{R}^n$ as follows $(x_1, x_2, \ldots, x_n) * (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$, so that it seems natural to look for some *multiplicative* real-valued order-preserving map to represent the preference.

Thus, it seems mandatory to prepare a new sets of results concerning the continuous numerical representability of total preorders defined on these "richer" structures, namely the semitopological real algebras, through real-valued order-preserving maps that are linear and multiplicative.

This is the objective of the present paper, whose structure goes as follows: After the Introduction and a section on previous definitions and notations, we study (in Section 3) conditions that characterize the existence of continuous real-valued order-preserving maps that are linear and multiplicative and represent total preorders defined on a semitopological real algebra. Finally, in Section 4 we add further theoretical results in this direction, but this time inspired by contexts coming from Social Choice theory.

2 Previous Definitions and Notations

Throughout the paper, $\precsim$ will denote a total preorder defined on a (nonempty) set X, i.e., a binary relation on X which is *reflexive* ($x \precsim x$ for all $x \in X$), *transitive* ($x \precsim y$, $y \precsim z$ implies $x \precsim z$), and *total* (either $x \precsim y$ or $y \precsim x$ for every $x, y \in X$). We also say that $(X, \precsim)$ is a *totally preordered set*. Two elements $x, y \in X$ are said to be *indifferent* if $x \precsim y$ and $y \precsim x$ (briefly, $x \sim y$). If, in addition, $\precsim$ is anti-symmetric ($x \precsim y$ and $y \precsim x$ implies $x = y$), then $\precsim$ is said to be a *total order*.

Given a total preorder $\precsim$, we consider as usual the *asymmetric binary relation* $\prec$ on X defined as: $x \prec y$ if and only if not ($y \precsim x$). The preorder $\precsim$ is said to be *non-trivial* if there are $x, y \in X$ such that $x \prec y$. We also consider the *symmetric binary relation* $\sim$ on X given by $x \sim y \iff (x \precsim y) \wedge (y \precsim x), \quad x, y \in X$.

Remark 2.1 In contexts coming mainly from applications in Economics or Social Choice, it is common to use the term "*preference*" to deal with a total preorder $\precsim$. As a matter of fact, in those contexts, a total preorder $\precsim$ is known as a *weak preference* (or *large preference*) whereas its associated asymmetric part $\prec$ is known as the *strict preference* and the symmetric part $\sim$ is called the associated *indifference*, which is an equivalence relation.

Let $(X, \precsim)$ be a totally preordered set. A real-valued function $u : X \to \mathbb{R}$ is said to be *order-preserving* (also known as a *utility function* or *numerical representation*) for $\precsim$ if, for every $x, y \in X$, it holds that $x \precsim y \iff u(x) \leq u(y)$. The total preorder $\precsim$ is said to be *representable* if it admits a real-valued order-preserving (utility) function. (Several characterizations of the representabiity of total preorders may be seen in [13], pp. 1–48.)

We now introduce two important definitions in the context of real vector spaces endowed with a total preorder.

In what follows $(X, +, \cdot)$, or simply X, will denote a real vector space. The zero vector in X will be denoted by $\mathbf{0}$.

Definition 2.2 A total preorder $\precsim$ defined on X is said to be:

(1) *translation-invariant* (see [4]), or *compatible with the operation* $+$, if $x \precsim y$ implies $x + z \precsim y + z$, $(x, y, z \in X)$,
(2) *homothetic* if $x \precsim y$ implies $\lambda \cdot x \precsim \lambda \cdot y$, $(x, y \in X, 0 \leq \lambda)$.

Definition 2.3 The structure $(X, \precsim, +, \cdot)$ is said to be a *totally preordered real vector space* if $(X, +, \cdot)$ is a real vector space and $\precsim$ is a total preorder on X that satisfies properties (1) and (2) above.

Definition 2.4 A function $v : X \longrightarrow \mathbb{R}$ is said to be *additive* if $v(x + y) = v(x) + v(y)$, $(x, y \in X)$, and it is said to be *linear* if $v(\lambda \cdot x + \beta \cdot y) = \lambda v(x) + \beta v(y)$, $(x, y \in X, \lambda, \beta \in \mathbb{R})$.

Let us now consider some *topological* properties of total preorders defined on a topological space. We introduce the most common notion of continuity in this context.

Definition 2.5 Let (Z, τ) be a topological space. A total preorder $\precsim$ defined on Z is said to be *continuous* if, for every $x \in Z$, the lower and the upper contour sets $L(x) = \{y \in Z;\ y \precsim x\}$ and $G(x) = \{y \in Z;\ x \precsim y\}$, respectively, are τ-closed subsets of Z.

Recall that a topological space (Z, τ) is *connected* if it cannot be written as the disjoint union of two non-empty open subsets. It is said to be *separable* if it contains a countable dense subset.

The next fundamental result will be used in the sequel. It provides topological conditions for a continuous total preorder defined on a topological space to admit a continuous real-valued order-preserving (utility) function. It is known as the Eilenberg representation theorem (see [25]).

Theorem 2.6 *Let (Z, τ) be a connected and separable topological space and let $\precsim$ be a continuous total preorder on Z. Then there is a continuous real-valued order-preserving (utility) function that represents $\precsim$.*

Let us come back to our algebraic setting $(X, +, \cdot)$. In order to consider topological properties of total preorders defined on X we need X to be endowed with a topology, say τ. Because it is our aim to obtain a particular representation (e.g., linear and continuous) of total preorders it is natural to assume some kind of relationship of *compatibility* between the operations $+$ and $\cdot$ and the topology τ. In particular, it will be assumed in the sequel that $(X, \tau, +, \cdot)$ is a *topological real vector space* (For an extensive treatment of the theory of topological vector spaces see, e.g., [40]).

Remark 2.7 The structure $(X, \precsim, \tau, +, \cdot)$ is said to be a *totally preordered topological real vector space* if, both, $(X, \precsim, +, \cdot)$ is a totally preordered real vector space and $(X, \tau, +, \cdot)$ is a topological real vector space. In other words, the total preorder $\precsim$ is compatible with the binary operation $(+)$ and the external operation $(\cdot)$, and the algebraic operations are continuous (considering on X the given topology τ, on the real line $\mathbb{R}$ the usual (Euclidean) topology and on $X \times X$ as well as on $\mathbb{R} \times X$ the corresponding product topologies).

3 Representation Theorems for Total Preorders on Topological Vector Spaces

We begin this section by recalling two existing results involving conditions (1) and (2) of Definition 2.2 above. Remember that a nonempty subset $K \subseteq X$ of a real vector space X is said to be a *real cone* if $\lambda k \in K$, for every vector $k \in K$ and every non-negative scalar $\lambda \geq 0 \in \mathbb{R}$. A function $v : K \longrightarrow \mathbb{R}$ is said to be *homogeneous* if $v(\lambda k) = \lambda v(k)$, $(k \in K, 0 \leq \lambda)$.

The following theorem provides a characterization of the representability of a total preorder defined on a cone of a topological real vector space by means of a continuous and homogeneous real-valued order-preserving (utility) function.

Theorem 3.1 *Let K be a real cone endowed with a total preorder $\precsim$ in a topological real vector space X. Then, there is a continuous and homogeneous utility function representing $\precsim$ if and only if $\precsim$ is homothetic and continuous.*

Proof See [9], Corollary on page 297. □

For the case of totally preordered topological real vector spaces the following characterization result is known.

Theorem 3.2 *Let X be a a topological real vector space endowed with a total preorder $\precsim$. Then, there is a linear and continuous real-valued order-preserving (utility) function representing $\precsim$ if and only if $\precsim$ is translation-invariant, homothetic and continuous.*[1]

Proof See [23], Theorem on page 521. □

Remark 3.3 It should be noted that the proofs of both results above are based on the solution of some *functional equations*; namely, the functional equation of homotheticity (namely, $f(x) = f(y) \iff f(tx) = f(ty)$ or equivalently $f(tx) = g(t, f(x))$,

[1] Actually this result, as shown in [23], still holds under weaker conditions. Continuity can be replaced by upper-semicontinuity at zero, which means that $\{x \in X; \mathbf{0} \precsim x\}$ is a closed subset of X. Homotheticity is then a consequence of upper-semicontinuity at zero and translation-invariance. It should be noted that this remark also applies to Theorem 3.14 below. (See also [12] for other results in this direction.)

for every $x, y \in X,\ t > 0 \in \mathbb{R}$, and g some function of two variables, see [18, 19, 27] for further details), and the functional equation of translation-invariance (namely, $f(x) = f(y) \iff f(x+z) = f(y+z)$, for every $x, y, z \in X$) respectively.

In addition to translation-invariance and homotheticity for $\precsim$, we wish to study the implications of introducing another invariance property, namely, that of homogeneity with respect to a third operation, the product operation. This means that our abstract real vector space X need to be equipped with an additional binary operation, say $*$. If there are $x, y \in X$ such that $x * y \neq \mathbf{0}$, then we will say that $*$ is non-zero. Otherwise, we will say that $*$ is zero (or trivial). We begin with a basic definition.

Definition 3.4 A *real algebra*[2] $(X, +, \cdot, *)$ is a space X endowed with three binary operations such that:

(i) $(X, +, \cdot)$ is a real vector space.
(ii) $(X, +, *)$ is a ring.[3]
(iii) $\lambda \cdot (x * y) = (\lambda \cdot x) * y = x * (\lambda \cdot y)$, for all $x, y \in X$, for all $\lambda \in \mathbb{R}$.

Let $(X, +, \cdot, *)$, or simply X, be a real algebra. A function $v : X \longrightarrow \mathbb{R}$ is said to be an *algebra-homomorphism* if it is linear and multiplicative (i.e., $v(x * y) = v(x)v(y)$, $(x, y \in X)$).

Now we introduce the concept of a multiplicative total preorder defined on a real algebra.

Definition 3.5 Let X be a real algebra. A total preorder $\precsim$ defined on X is said to be *multiplicative*[4] if $x \precsim y$, implies $z * x \precsim z * y$ and $x * z \precsim y * z$, for all $x, y, z \in X$ such that $\mathbf{0} \precsim$ z.

Remark 3.6 A related concept was introduced, for the particular case of $\mathbb{R}^n$, in [20], Definition 4.6.

Next, we introduce the definition of a totally preordered real algebra.

Definition 3.7 A *totally preordered real algebra*[5] $(X, \precsim, +, \cdot, *)$, or simply X, is a real algebra $(X, +, \cdot, *)$ equipped with a translation-invariant, homothetic and multiplicative total preorder $\precsim$.

[2]The reader will find the textbook of Hungerford [31] a useful reference for algebraic issues.

[3]Remember that a *ring* is a nonempty set R together with two binary operations (usually denoted as addition (+) and multiplication) such that $(R, +)$ is a group, and the following properties hold for every $a, b, c \in R$: $a + b = b + a$, $(ab)c = a(bc)$, and $a(b + c) = ab + ac$. (See e.g. [31], pp. 115 and ff.)

[4]If $*$ is zero, i.e., $x * y = \mathbf{0}$, for every $x, y \in X$, then every total preorder on X is trivially multiplicative.

[5]If a ring is equipped with a translation-invariant and multiplicative total preorder, then we reach the notion of a totally preordered ring.

If we consider now real algebras endowed with a *topology*, then we reach the following key definition.

Definition 3.8

(i) A *semitopological real algebra* $(X, \tau, +, \cdot, *)$ is a real algebra $(X, +, \cdot, *)$ such that $(X, \tau, +, \cdot)$ is a topological vector space.
(ii) A *totally preordered semitopological real algebra* $(X, \precsim, \tau, +, \cdot, *)$, or simply X, is a topological vector space $(X, \tau, +, \cdot)$ endowed with both a total preorder $\precsim$ and a binary operation $*$ such that $(X, \precsim, +, \cdot, *)$ is a totally preordered real algebra.

Remarks 3.9

(i) Notice that, in the previous Definition 3.8 (i) no assumptions involving topological considerations are made on $*$. If $*$ is also a τ-continuous binary operation, then the structure $(X, \tau, +, \cdot, *)$ is said to be a *topological real algebra*.
(ii) The structure $(X, \precsim, \tau, +, \cdot, *)$ is said to be a *totally preordered topological real algebra* (respectively, a *totally preordered semitopological real algebra*) if $(X, \tau, +, \cdot, *)$ is a topological (respectively, semitopological) real algebra and $(X, \precsim, +, \cdot, *)$ is a totally preordered real algebra.
(iii) It should be pointed out that, sometimes in the literature and unlike our notion, the concept of a semitopological real algebra requires the operation $*$ to be separately continuous (that is, the structure $(X, \tau, +, \cdot, *)$ satisfies that for any fixed $a \in X$ it holds that the map $p : X \to X$ given by $p(x) = a * x, \quad x \in X$ is τ-continuous. See, e.g., [42]). By the way, in this case the nomenclature "topological real algebra" is kept for a structure $(X, \tau, +, \cdot, *)$ such that the operation $*$ is jointly continuous (that is, the map $q : X \times X \to X$ given by $q(x, y) = x * y, \quad x, y \in X$ is continuous considering on X the topology τ and on $X \times X$ the corresponding product topology $\tau \times \tau$).

Within this formal context, we can state the concept of a straight total preorder.

Definition 3.10 Let X be a semitopological real algebra. A total preorder $\precsim$ defined on X is said to be *straight* if there is a continuous real-valued order-preserving (utility) function for $\precsim$ which is an algebra-homomorphism.

Remark 3.11 We use here the term "*straight*" because $\precsim$ actually plays a role similar to the usual Euclidean order $\leq$ on the straight (real) line $\mathbb{R}$ endowed with its usual binary operations of addition or sum $(+)$ and multiplication $(\cdot)$. It is plain that $(\mathbb{R}, +, \cdot)$ is indeed a topological real algebra.

Before presenting our main result in this framework, we need a preliminary lemma.

Lemma 3.12 *Let $*$ be a binary operation defined on $\mathbb{R}$ and let $\precsim$ be a continuous total order such that $(\mathbb{R}, \precsim, +, *)$ is a totally ordered ring (i.e. $(\mathbb{R}, +, *)$ is a ring and $\precsim$ is translation-invariant and multiplicative).*

(i) *If $*$ is non-zero, then $\precsim$ has a continuous representation which, in addition, is additive and multiplicative.*
(ii) *If $*$ is zero, then $\precsim$ has a continuous and additive representation.*

Moreover, if $*$ is non-zero, then there exists $a \in \mathbb{R} \setminus 0$ such that $x * y = axy$, for all $x, y \in \mathbb{R}$.

Proof Since $\precsim$ is continuous on $\mathbb{R}$ it follows, by the Eilenberg representation theorem, that there exists a continuous real-valued order-preserving (utility) function $u : \mathbb{R} \longrightarrow \mathbb{R}$ for $\precsim$. Moreover, since $\precsim$ is a total order, u is injective (actually, it is an homeomorphism). This, in particular, implies that $(\mathbb{R}, \precsim, +)$ is *Archimedean* (i.e., if $0 \prec a \prec b$, then there is $n \in \mathbb{N}$ such that $b \prec na$). Now, by a result of Pickert and Hion (see [29], Th. 1, p. 126), either $*$ is zero and then there is an additive numerical representation of $\precsim$, say w, or $\precsim$ has a numerical representation, say v, which is additive and multiplicative. Then, since u is continuous and injective, it is either a strictly increasing or a strictly decreasing function. Assume that it is strictly increasing, the other case being analyzed similarly. Suppose that $*$ is non-zero. Then, since u and v are real-valued order-preserving (utility) functions for $\precsim$, v is also strictly increasing. Now, remember that v is an additive function and it is well known that if v is discontinuous at some point, then it is discontinuous at every point of $\mathbb{R}$ (see e.g. [7], pp. 125 and ff.). This last fact, v being increasing, leads to $v(x) = ax$ (with $a > 0$). If $*$ is zero, then the same argument as above leads to the same kind of representation for $\precsim$, i.e., $w(x) = ax$ (with $a > 0$) and we are done. To prove the last assertion of the statement of Lemma 3.12 notice that, since $*$ is non-zero, for every $x, y \in \mathbb{R}$, it holds that $v(x * y) = a(x * y) = v(x)v(y) = a^2xy$. Therefore, $x * y = axy$ for every $x, y \in \mathbb{R}$ and the proof is finished. □

Remark 3.13 If we use semicontinuity instead of continuity it could be possible to consider order-preserving functions that take values on the set of *rational* numbers $\mathbb{Q}$. (See [15] for details).

We are now ready to introduce our main characterization result.

Theorem 3.14 *Let X be a semitopological real algebra endowed with a total preorder $\precsim$.*

(i) *If $*$ is non-zero, then $\precsim$ is straight if and only if it is continuous, translation-invariant, homothetic and multiplicative.*
(ii) *If $*$ is zero, then $\precsim$ has a linear and continuous numerical representation if and only if it is continuous, translation-invariant and homothetic.*

Proof It is straightforward to prove the "only if" part of statements (i) and (ii). In order to prove the "if" part, we first consider the case that $*$ is non-zero. Thus, we have to show that there is a continuous, linear and multiplicative representation $\psi : X \longrightarrow \mathbb{R}$ for $\precsim$. If $\precsim$ is trivial (i.e., $x \sim y$, for every $x, y \in X$), then $\psi \equiv 0$ works. Suppose then that $\precsim$ is non-trivial and consider the set $I(\mathbf{0}) = \{\mathrm{x} \in \mathrm{X};\ \mathrm{x} \sim \mathbf{0}\}$. First, let us see that $I(\mathbf{0})$ is both a vector subspace and an ideal[6] of X, for which we need to prove the following two properties:

(a) $I(\mathbf{0})$ is a real vector subspace of X.
(b) For every $x \in I(\mathbf{0})$, $y \in X$, it holds that $x * y \in I(\mathbf{0})$ and $y * x \in I(\mathbf{0})$.

Let $x, y \in I(\mathbf{0})$. Because $\precsim$ is translation-invariant, it follows that $x + y \sim x + \mathbf{0} \sim x \sim \mathbf{0}$. So, in order to prove (a), it is sufficient to see that, given $x \in I(\mathbf{0})$ and $\lambda \in \mathbb{R}$, then $\lambda \cdot x \sim \mathbf{0}$. If $\lambda \geq 0$ then, by homotheticity, $\lambda \cdot x \sim \mathbf{0}$. If $\lambda < 0$, then $(-\lambda) \cdot x \sim \mathbf{0}$. But $(-\lambda) \cdot x = -\lambda \cdot x$ and, by translation invariance of $\precsim$, $\lambda \cdot x \sim \mathbf{0}$.

To prove (b), let $y \in X$ and $x \in I(\mathbf{0})$. If $\mathbf{0} \precsim y$ then, since $\precsim$ is multiplicative, $x * y \precsim \mathbf{0} * y = \mathbf{0}$ and $\mathbf{0} = \mathbf{0} * y \precsim x * y$. Therefore, $x * y \sim \mathbf{0}$. If $y \precsim \mathbf{0}$, then $\mathbf{0} \precsim -y$ and so $-x * y \sim \mathbf{0}$. Because $x * y = -(-x) * y$, and $I(\mathbf{0})$ is a vector subspace of X, it holds that $x * y \sim \mathbf{0}$. The case $y * x$ is similar. Thus (b) holds and therefore $I(\mathbf{0})$ is an ideal of X.

Now, consider the quotient space $X/I(\mathbf{0})$ and denote it by Q. Because Q coincides with the quotient space $X/\sim$, Q is a totally ordered set. We denote the natural total order on Q by $\precsim'$. Since $I(\mathbf{0})$ is both a vector subspace and an ideal of X, the operations $+$, $\cdot_{\mathbb{R}}$ and $*$ pass to the quotient space in such a way that $\precsim'$, defined on Q, is also translation-invariant, homothetic, and multiplicative. Consider on Q the quotient topology, denoted by τ', and remember that $(Q, \tau', +, \cdot)$ is a topological vecor space which is separated (or Hausdorff) since $I(\mathbf{0})$ is a closed subset of X because $\precsim$ is continuous (see, e.g., [40]). Moreover, $\precsim'$ is clearly a τ'-continuous total order on Q. Let us observe that $(Q, \tau', +, \cdot)$ is homeomorphic, and isomorphic as a vector space, to $\mathbb{R}$ endowed with the usual Euclidean topology and ordinary binary operations. Suppose, by way of contradiction, that this claim is not true. If this the case, then the algebraic dimension of Q is greater than 1. Denote by $S \subseteq Q$ a two-dimensional subspace of Q. Then, it is well-known that the restriction of τ' to S induces the Euclidean topology on S (see, e.g., [2], Theorem 5.65). In particular, S is a connected and separable topological space endowed with a continuous total order $\precsim' |_S$, the restriction of $\precsim'$ to S. Then, by the Eilenberg representation theorem, there is a continuous real-valued order-preserving (utility) function, say $v : S \to \mathbb{R}$, that represents $\precsim' |_S$. Notice that such a function is injective since $\precsim' |_S$ is total. But it is not possible since, for a given $s \in S$, the set $S \setminus \{s\}$ is connected in S and $v(S \setminus \{s\})$ is not connected in $\mathbb{R}$, which contradicts the continuity of v. So, Q is homeomorphic, and isomorphic as a vector space, to the reals. Now, by Lemma 3.12, there is a continuous representation $u : Q \to \mathbb{R}$ for $\precsim'$, which is linear and multiplicative. Let us denote by $p : X \to Q$ the projection map. It is easy to see that p is linear, multiplicative and continuous. Then, by considering the composition $\psi = u \circ p : X \to \mathbb{R}$ we obtain a continuous, linear and multiplicative real-valued order-preserving (utility) function which represents $\precsim$ and therefore we have proved statement (i).

Suppose now that $*$ is zero. In this case we have to show that there is a linear and continuous representation $\psi : X \longrightarrow \mathbb{R}$ for $\precsim$. If $\precsim$ is trivial, then $\psi \equiv 0$ works. Otherwise, we argue exactly as above by considering the set $I(\mathbf{0})$. Notice that now $I(\mathbf{0})$ is just a closed real vector subspace of X. Following the same reasoning as above (notice that Lemma 3.12 also covers the case in which $*$ is zero), we obtain a linear

[6]Let $(X, +, *)$ be a ring. A subset $I \subseteq X$ is said to be an ideal of X if $(I, +)$ is an additive subgroup of X and $a * x, x * a \in I$, for every $a \in I, x \in X$.

and continuous representation $u : Q \to \mathbb{R}$ for $\precsim'$. The desired linear and continuous representation ψ for $\precsim$ is obtained by taking the composition of u with the projection map $p : X \to Q$. □

As a consequence of the previous theorem the following information can be added.

Corollary 3.15 *If X is a Banach algebra and $*$ is non-zero, then the set of all translation-invariant, homothetic, multiplicative and continuous total preorders on X can be identified with the spectrum of X.*[7]

Remarks 3.16

(i) Statement (i) of Theorem 3.14 above can be rephrased as follows. Let $(X, \tau, +, \cdot, *)$ be a semitopological real algebra and let $\precsim$ be a total preorder on X. Then, $(X, \precsim, \tau, +, \cdot, *)$ is a totally preordered semitopological real algebra if and only if $\precsim$ is straight.
(ii) Theorem 3.14 can be understood as a natural generalization of Theorem 3.2 above.

We now provide some application of the previous theorem to the context of infinite-dimensional spaces with a Schauder basis. First, let us state some definitions.

Definition 3.17 Let $(X, \tau, +, \cdot)$ a topological real vector space. A sequence $(e_j)_{j=1}^{\infty} \subset X$ of (finitely) linearly independent vectors of X is said to be a *Schauder basis* if, for every $x \in X$, there are numbers $(x_j)_{j=1}^{\infty} \subset \mathbb{R}$ such that $x = \sum_{j=1}^{\infty} x_j e_j = lim_{n\to\infty} \sum_{j=1}^{n} x_j e_j$.

Remarks 3.18

(i) It sould be noted that a topological real vector space with a Schauder basis is necessarily *separable*. (See e.g. [2], p. 498).
(ii) It is very simple to see that if $(e_j)_{j=1}^{\infty}$ is a Schauder basis of X then for every $x \in X$ the coefficients $(x_j)_{j=1}^{\infty}$ are uniquely determined. So, for every $x \in X$, we can write $x = (x_j)$.

Definition 3.19 A Schauder basis $(e_j)_{j=1}^{\infty}$ of a semitopological real algebra $(X, \tau, +, \cdot, *)$ is said to be *multiplicative* if $e_j * e_i$ is equal to e_j whenever $i = j$, and it is equal $\mathbf{0}$ otherwise.

Remarks 3.20

(i) If $(e_j)_{j=1}^{\infty}$ is a multiplicative Schauder basis of X then, for every $j \in \mathbb{N}$, the linear operator $x \in X \to \psi(x) = x_j e_j \in X$ is multiplicative and conversely.

[7] The spectrum of a Banach algebra is defined as the set of all its multiplicative linear functionals. For details about Banach algebras, see [39]. Some results on real-valued order-preserving (utility) functions on Banach spaces and, in particular, Banach algebras may be seen in [14].

(ii) If $(e_j)_{j=1}^{\infty}$ is a multiplicative Schauder basis of a topological real algebra X then, for every $x, y \in X$, $x * y = \sum_{j=1}^{\infty} x_j y_j e_j$.

We then reach the following result.

Theorem 3.21 *Let $(X, \tau, +, \cdot, *)$ be a semitopological real algebra with a multiplicative Schauder basis $(e_j)_{j=1}^{\infty}$ and let $\precsim$ be a non-trivial total preorder defined on X. Then the following assertions are equivalent:*

(i) *$\precsim$ is translation-invariant, homothetic, multiplicative and continuous,*
(ii) *$\precsim$ is straight,*
(iii) *There is $i \in \mathbb{N}$, such that, for every $x = (x_j)$, $y = (y_j) \in X$, it holds that $x \precsim y$ if and only if $x_i \leq y_i$.*

Proof (i) implies (ii) follows from Theorem 3.14. (iii) implies (i) is routine. So, it remains to prove (ii) implies (iii). To that end, let $\psi : X \to \mathbb{R}$ be a continuous, linear and multiplicative real-valued order-preserving (utility) function that represents $\precsim$. Since ψ is linear and continuous, it holds that $\psi(x) = \sum_{j=1}^{\infty} x_j \psi(e_j)$, $(x \in X)$. Notice that all, but at most one, numbers $\psi(e_j)$ are zero. Indeed, suppose by way of contradiction that there exist $i \neq j$ such that $\psi(e_i)\psi(e_j) \neq 0$. Then, because ψ is multiplicative as well as the basis (e_j) is, $0 = \psi(\mathbf{0}) = \psi(e_i * e_j) = \psi(e_i)\psi(e_j)$, which is a contradiction. So, there is $i \in \mathbb{N}$ such that, for every $x \in X$, $\psi(x) = x_i \psi(e_i)$. Moreover, since $\psi(e_i^2) = \psi(e_i)\psi(e_i) = \psi(e_i)$ and $\precsim$ is non-trivial, it follows that $\psi(e_i) = 1$. Therefore, $\psi(x) = x_i$, for every $x \in X$, and we are done. □

4 Further Results Inspired by Contexts Related to Social Choice

In this section we intend to explain, interpret and reinforce the results obtained in the formal framework considered in the previous section, as well as to obtain new ones, having in mind some key contexts that arise in the Social Choice theory framework. Although we introduce in this section a new set of pure mathematical results, we motivate them by explaining how the inspiration can be encountered in some Social Choice contexts, and, in addition, to what extent these new results obtained are useful to deal with important concepts coming from Economics, Social Choice and Decision-Making.

Thus, one of the most important issues in the Social Choice theory has to do with the problem of aggregating individual judgments or opinions into a social one. To be more precise, suppose that a society with n individuals ($n \geq 2$) is given. Suppose in addition that each individual has a (preference) ranking over a set of m candidates ($m \geq 2$), say X. The preference of each individual turns out to be a total preorder over X. Denote the set of all total preorders, or preferences, defined on X by P. The aggregation problem consists in finding functions $F : P^n = P \times \cdots \times P_{n-times} \to P$ which satisfy certain properties of, say, common sense. For a given profile of individual preferences $(p_j) = (p_1, \cdots, p_n) \in P^n$, the element $F((p_j))$, or to make the notation easier just $F(p_j)$, is interpreted as the social preference associated to it. Also, for a given $p \in P$, p_s will denote the asymmetric binary relation, called the *strict preference*, associated to p. Similarly, for a given profile $(p_j) \in P^n$, $F(p_j)_s$ will denote the asymmetric binary relation associated to $F(p_j)$.

Two typical (and classical in the Social Choice literature) properties for an aggregation function to be satisfied are the following ones:

(i) *Pareto property*: For any preference profile $(p_j) \in P^n$ and any pair of alternatives $x, y \in X$ if $xp_{j\,s}y$, for all $j \in \{1, \cdots, n\}$, then $xF(p_j)_s y$,
(ii) *Independence of irrelevant alternatives condition*: For any pair of preference profiles (p_j), $(q_j) \in P^n$ and any pair of alternatives $x, y \in X$ such that for all $j \in \{1, \cdots, n\}$, $xp_j y \Longleftrightarrow xq_j y$ and $yp_j x \Longleftrightarrow yq_j x$, it follows that $xF(p_j)y \Longleftrightarrow xF(q_j)y$ and $yF(p_j)x \Longleftrightarrow yF(q_j)x$.

Roughly speaking, Pareto property says that if, for a given profile $(p_j) \in P^n$, all individuals (strictly) prefer candidate a to candidate b, then the social preference associated to that profile, $F(p_j)$, also (strictly) ranks a over b. The independence of irrelevant alternatives condition states that the way in which society ranks two arbitrary candidates only depends upon the way in which individuals rank these two candidates and nothing else.

This conceptual framework is called the Arrovian model in social choice theory since [3]. Quite surprisingly, Arrow proved that if an aggregation function F with $n \geq 2$ individuals and $m \geq 3$ candidates satisfies conditions (i) and (ii) above, then it is dictatorial, which, roughly speaking again, means that there is an individual in the society so that, in order to (strictly) rank the candidates, only her (his) opinion is taken into account.

In a related context Sen [41] introduced the concept of a social welfare functional. Suppose that each individual preference can be represented by an order-preserving real-valued function (called in this literature, a utility function). Let us denote by U the set of all (utility) functions defined on X. Then, any aggregation function $F : P^n \to P$ induces in a natural way a so-called social welfare functional $G : U^n \to P$ by letting $G((u_1, \cdots, u_n)) = F((p_1, \cdots, p_n))$, where, for every $i \in \{1, \cdots, n\}$, $p_i \in P$ is the preference relation associated to u_i. It is very easy to see that G is well-defined. What is really important in this approach is that if $n \geq 2$, $m \geq 3$ and F satisfies conditions (i) and (ii) of the Arrovian model, then there is a total preorder, say $\precsim$, defined on $\mathbb{R}^n$ so that all the information given by G is conveyed by $\precsim$ (for details, see [30]). So, from this perspective, for a given total preorder $\precsim$ defined on $\mathbb{R}^n$, or other spaces with interpretation in Economics, it is interesting to obtain conditions that characterize when the preorder $\precsim$ is dictatorial.

In this sense, we provide in this section an algebraic characterization of this fact. In addition, we also offer new insights of this theory in a more general context involving infinite-dimensional sequence spaces.

Definition 4.1 A total preorder $\precsim$ defined on $\mathbb{R}^n$ is said to be *one-dimensional* (or *strongly dictatorial*) if there is $i \in \{1, 2, \cdots, n\}$, such that, for every $x = (x_j)$, $y = (y_j) \in \mathbb{R}^n$, it holds that $x \precsim y$ if and only if $x_i \leq y_i$.

Remarks 4.2 In a mathematical context, we would use the nomenclature "*one-dimensional*" because the preorder $\precsim$ despite being defined on $\mathbb{R}^n$ for some n that is (usually) bigger than 1, acts as if it were defined only on just *one* coordinate. Nevertheless, in contexts coming from Social Choice the terminology "*strongly dictatorial*" seems to be more suitable and, as a matter of fact, common.

Let us consider $\mathbb{R}^n$ endowed with the usual operations $+$, $\cdot$ and $*$, defined coordinatewise, and the usual Euclidean topology τ. Then we reach the following characterization of those total preorders which are one-dimensional (strongly dictatorial).

Theorem 4.3 *Let $\precsim$ be a non-trivial total preorder defined on $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) *$\precsim$ is translation-invariant, homothetic, multiplicative and continuous,*
(ii) *$\precsim$ is straight,*
(iii) *$\precsim$ is one-dimensional (strongly dictatorial).*

Proof It is a direct consequence of Theorem 3.21 above. Notice that the canonical basis of $\mathbb{R}^n$ is a multiplicative Schauder basis. □

Remarks 4.4

(i) The previous result can be restated as follows: The only total preorders that make $(\mathbb{R}^n, \tau, +, \cdot, *)$ into a totally preordered topological real algebra are the one-dimensional (strongly dictatorial) ones.
(ii) Theorem 4.3 also applies to the infinite-dimensional spaces c_0 and l_1 (see e.g. [2], pp. 492–495) endowed with the usual operations defined coordinatewise. As in the Euclidean case, the multiplicative Schauder basis is the corresponding canonical basis. The c_0 space consists of all real sequences which vanish at infinity equipped with the topology given by the supremum norm ($\|(x_n)\|_\infty = sup\{|x_n|; n \in \mathbb{N}\}$). It is easy to show that c_0 is a semitopological real algebra (in fact, it is a (Banach) topological real algebra). The space l_1 consists of all real sequences (x_n) such that $\sum |x_n| < \infty$. Equipped with the topology given by the norm defined as $\|(x_n)\|_1 = \sum |x_n|$, l_1 is a semitopological real algebra (in fact, it is a (Banach) topological real algebra). Both spaces are usually encountered in the economics literature related to general equilibrium theory in infinite-dimensional context (see, e.g., [5]).
(iii) The existence of a multiplicative Schauder basis is crucial for the previous theorem to hold. If such a basis does not exist then the conclusion can fail. This claim occurs even though the space has a sequence structure. For instance, consider the space $X = l_\infty$ which consists of all bounded real sequences. This space naturally arises in intertemporal decision problems involving an infinite horizon (see, e.g., [34]). Endowed with the usual operations defined coordinatewise and the supremum norm, l_∞ is a semitopological real algebra (in fact, it is a Banach real algebra). In this case, the set of straight total preorders strictly contains the set of strong dictatorial ones (i.e., those which are defined as the projection over the corresponding coordinate). To better illustrate this assertion, let us denote by $\beta(\mathbb{N})$ the Stone-Čech compactification of the set of the natural numbers $\mathbb{N}$ (see, e.g., [24]). Let $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Then the evaluation map at p defines a linear and multiplicative map $x \in l_\infty \rightsquigarrow e_p(x) \in \mathbb{R}$. Moreover, since $(l_\infty, \|\cdot\|_\infty, +, \cdot, *)$ is a Banach algebra, it is well known that every linear and multiplicative real-valued function is continuous (see, e.g., [39]). Thus, the order $\precsim_p$ defined on l_∞ as $x \precsim_p y \Leftrightarrow e_p(x) \leq e_p(y)$ is straight (and continuous). However, it is not a projection over any coordinate. It should be noted that, from the point of view of an economic interpretation,

$\precsim_p$ suggests the presence of an *invisible dictator* (we mean, straight preorders which are not one-dimensional (strongly dictatorial)). As far as we know the concept of an invisible dictator was first introduced in [33]. On the one hand, in the context of Kirman and Sonderman paper, invisible dictators are associated to *free ultrafilters*[8] on $\mathbb{N}$. On the other hand, it is well-known (see [24]) that the set $\beta(\mathbb{N}) \setminus \mathbb{N}$ can be identified with the set of free ultrafilters on $\mathbb{N}$. Moreover, it is also known in the theory of Banach algebras, that the set of (continuous), linear and multiplicative real-valued functions defined on l_∞ is $\beta(\mathbb{N})$ (see, e.g., [39]). So, these existing links are the reason why we call $\precsim_p$, $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$, an invisible dictator.

Another interesting invariance property, involving the usual product operation $*$ in $\mathbb{R}^n$, that appears in the economics literature is introduced in the following definition. Given two vectors $x = (x_i)$, $y = (y_i) \in \mathbb{R}^n$, $x \leq y$ stands for the usual partial order in $\mathbb{R}^n$ (i.e., $x \leq y$ iff $x_i \leq y_i$, for all $i \in N$). In a similar way, "$\ll$" will stand for the strict partial order in $\mathbb{R}^n$ (i.e., $x \ll y$ iff $x_i < y_i$, for all $i \in N$).

Definition 4.5 A total preorder $\precsim$ in $\mathbb{R}^n$ is said to be *scale-independent* if $x \precsim y$, $\mathbf{0} \ll \mathrm{z}$ implies $z * x \precsim z * y$, $(x, y, z \in \mathbb{R}^n)$.

Roughly speaking, this condition means that particular reparametrizations of the individual utilities do not change the preference relation. In other words, $\precsim$ is invariant to independent changes of units in which, say, utility is measured. This condition appears also in the theory of axiomatic bargaining to characterize Nash collective social choice functions, (see [36]), as well as in the literature on interpersonal comparability in Social Choice.

We now prove that both, scale-independence and the multiplicative property, are very close to one another. First we need to introduce a definition.

Definition 4.6 A total preorder $\precsim$ in $\mathbb{R}^n$ is said to be *increasing* if $x \leq y$ implies $x \precsim y$, for all $x, y \in \mathbb{R}^n$.

Theorem 4.7 *Let $\precsim$ be a total preorder defined on $\mathbb{R}^n$. Assume that $\precsim$ is translation-invariant and continuous. Then the following assertions are equivalent:*

(i) *$\precsim$ is multiplicative with respect to the usual coordinatewise product $*$ in $\mathbb{R}^n$.*
(ii) *$\precsim$ is scale-independent and increasing.*

Proof Suppose that $\precsim$ is non-trivial, since otherwise the result is obvious. Assume then that it is multiplicative. Then, by Theorem 4.3, $\precsim$ is one-dimensional (strongly dictatorial) and it follows easily that $\precsim$ is scale-independent and increasing. For the converse, suppose that $\precsim$ is scale-independent and increasing. Then, since $\precsim$ is translation-invariant and continuous, by Theorem 3.14 (ii) there is a numerical representation ψ for $\precsim$ given by $\psi(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_j x_j$, for some $a_j \in \mathbb{R}$, $j = 1, \ldots, n$. It should be noted that $a_j \geq 0$, for all j because $\precsim$ is increasing. Let us see that all, but at most one, coefficients a_j are zero. Suppose, by way of contradiction,

[8]For information about ultrafilters and related items see e.g. [2] pp. 31 and ff.

that there exist $r \neq s$ such that $a_r a_s \neq 0$. Consider the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, defined as: $x_r = a_s, x_s = -a_r$ and $x_j = 0$ otherwise. Note that $\psi(x) = 0$. Now, by scale-independence, $\psi(z * x) = 0$, for every $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ such that $0_n \ll z$. But this would imply that $a_r a_s(z_r - z_s) = 0$, for any such z. Choosing a suitable z such that $z_r \neq z_s$, we easily conclude that either $a_r = 0$ or $a_s = 0$, which is a contradiction. Hence, there is $i \in \{1, \ldots, n\}$ such that $\psi(x_1, \ldots, x_n) = a_i x_i$. In addition, since $\precsim$ is non-trivial and increasing, it follows that $a_i > 0$; hence $\precsim$ is one-dimensional (strongly dictatorial). The result then follows from Theorem 4.3 again. □

Remarks 4.8 The assumption that $\precsim$ is increasing cannot be omitted from the statement of the previous result. Indeed, consider the total order $\precsim$ on $\mathbb{R}$ defined as $x \precsim y$ if and only if $-x \leq -y$. It is very easy to see that $\precsim$ so-defined is translation-invariant, homothetic, continuous and scale-independent. However, it is not increasing. Note that it is not multiplicative since $0 \prec -1$ and $1 = -1 * -1 \prec 0$.

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