Some results on duality of spaces of vector-valued Lipschitz functions

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Lipschitz functions and free spaces

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In particular, we denote $Lip(M) := Lip(M, \mathbb{R})$. For each $m \in M$, consider the evaluation functional $\delta_m \in Lip(M)$ given by $\delta_m(f) := f(m)$. The space

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- $\mathcal{F}(\mathbb{N}) = \ell_1$ $(\delta_n \mapsto e_1 + \ldots + e_n)$.
- $\mathcal{F}(\mathbb{R}) = L_1$ $(\delta_x \mapsto \chi_{(0,x)})$.

Lipschitz functions and free spaces

Let $f\colon M \to N$ be a Lipschitz map. Then there exists an operator $T : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $||T_f|| = ||f||_{Lip}$ and the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \delta\downarrow & & \delta\downarrow \\ \mathcal{F}(M) & \xrightarrow{T_f} & \mathcal{F}(N) \end{array}$$

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As a consequence $Lip(M, X)$ is isometric to $L(\mathcal{F}(M), X)$. In particular we get $Lip(M) = \mathcal{F}(M)^*$.

Duality in free spaces

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Notice that if M is separable and metrically convex then $\mathcal{F}(M)$ is a separable space without the RNP. Thus, it is not isomorphic to a dual Banach space.

Duality in free spaces

A natural candidate for a predual of $\mathcal{F}(M)$ is the so-called space of little-Lipschitz functions:

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Moreover, Dalet showed in 2015 that if M is a proper metric space (i.e. closed balls are compact sets), then

$$S(M) = \{f \in lip(M) : \lim_{r \to \infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} = 0\}$$

is a predual of $\mathcal{F}(M)$ whenever M is countable or ultrametric (i.e. $d(x,y) \leq \max\{d(x,z), d(y,z)\}$ for every $x, y, z \in M$).

Duality in free spaces

Another duality result is the following. We shall denote $lip_\tau(M) = lip(M) \cap C(M, \tau)$.

Theorem (Kalton,2004)

Let M be a separable complete bounded metric space. Suppose τ is a metrizable topology on M so that (M, τ) is compact and for every $x, y \in M$ and $\varepsilon > 0$ there exists $f \in lip_\tau(M)$ with $||f||_{Lip} \leq 1$ and $f(y) - f(x) \geq d(x, y) - \varepsilon$. Then the space $lip_\tau(M)$ is a predual of $\mathcal{F}(M)$.

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Corollary (Kalton, 2004)

Let X be a separable Banach space and $0 < \alpha < 1$. Then $lip_{\omega^*}(B_{X^*}, \| \|^\alpha)$ is a predual of $\mathcal{F}(B_{X^*}, \| \|^\alpha)$.

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Duality in vector-valued free spaces

Vector-valued Lipschitz free spaces were introduced by Becerra, López and Rueda in 2015 as

$$\mathcal{F}(M,X) := \overline{span}\{\delta_{m,x} : m \in M, x \in X\} \subseteq Lip(M,X^*)^*$$

where $\delta_{m,x}(f) := f(m)(x)$ for every $m \in M, x \in X$ and $f \in Lip(M,X^*)$.

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For which metric spaces M and Banach spaces X does there exists a subspace S of $Lip(M,X^{**})$ such that S is a predual of $\mathcal{F}(M,X^*)$?

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We will consider the spaces:

$$lip(M,X) := \left\{ f \in Lip(M,X) : \lim_{\varepsilon\to 0} \sup_{0<d(x,y)<\varepsilon} \frac{\|f(x)-f(y)\|}{d(x,y)} = 0 \right\},$$

$$S(M,X) := \left\{ f \in lip(M,X) : \lim_{r\to\infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x\neq y}} \frac{\|f(x)-f(y)\|}{d(x,y)} = 0 \right\}.$$

Duality in vector-valued free spaces

Let M be a proper metric space and X be a Banach space. Then $S(M,X)$ is a predual of $\mathcal{F}(M,X^*)$ in the following cases:

1. M is the middle third Cantor set.
2. M is countable.
3. M is ultrametric.
4. The metric is of the form d^α, where $0<\alpha<1$, and either $\mathcal{F}(M)$ or X^* has the AP.

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Our approach is strongly inspired in a paper by Jiménez-Vargas, Sepulcre and Villegas-Vallecillos. The main idea is to get an identification

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and thus

$$S(M,X)^* = (S(M)\widehat{\otimes}_\varepsilon X)^* = S(M)^*\widehat{\otimes}_\pi X^* = \mathcal{F}(M)\widehat{\otimes}_\pi X^* = \mathcal{F}(M,X^*).$$

Duality in vector-valued free spaces

Theorem (Kalton,2004)

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Let M be a pointed metric space and let τ be a topology on M such that (M,τ) is compact and d is τ-lower semicontinuous. Then

- $lip_\tau(M,X)$ is isometrically isomorphic to $K_{w^*,w}(X^*, lip_\tau(M))$.
- If either $lip_\tau(M)$ or X has the AP, then $lip_\tau(M,X)$ is isometrically isomorphic to $lip_\tau(M)\widehat{\otimes}_\varepsilon X$.
- If the assumptions of Kalton's result hold and either $\mathcal{F}(M)$ or X^* has the AP, then $lip_\tau(M,X)$ is a predual of $\mathcal{F}(M,X^*)$.

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Almost squareness

Definition (Abrahamsen-Langemets-Lima, 2016)

A Banach space X is said to be *almost square* (ASQ) if for every $x_1, \ldots, x_k \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that

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Roughly speaking, we can say that ASQ Banach spaces have a strong c_0 behaviour from a geometrical point of view.

Does there exist a dual ASQ Banach space?

We will give a partial answer to the above question by using the notion of almost squareness.

Unconditional almost squareness

Definition

Let X be a Banach space. We will say that X is *unconditionally almost square (UASQ)* if, for each $\varepsilon > 0$, there exists a subset $\{x_\gamma\}_{\gamma \in \Gamma} \subseteq S_X$ such that

1. For each $\{y_1, \ldots, y_k\} \subseteq S_X$ and $\delta > 0$ the set

$$\{\gamma \in \Gamma : \|y_i \pm x_\gamma\| \leq 1 + \delta \ \forall i \in \{1, \ldots, k\}\}$$

   is non-empty.
2. For every F finite subset of Γ and every choice of signs $\xi_\gamma \in \{-1, 1\}$, $\gamma \in F$, it follows $\|\sum_{\gamma \in F} \xi_\gamma x_\gamma\| \leq 1 + \varepsilon$.

Unconditional almost squareness

Example

1. The space $c_0(\Gamma)$ is UASQ.
2. $\ell_\infty^c(\Gamma) := \{x \in \ell_\infty(\Gamma) : supp(x) \text{ is countable}\}$ is UASQ whenever Γ is uncountable.
3. Given Γ an infinite set and $\mathcal{U}$ a free ultrafilter over Γ, the space $X := \{x \in \ell_\infty(\Gamma) : \lim_{\mathcal{U}}(x) = 0\}$ is UASQ.

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Let X be a separable Banach space. If X is ASQ, then X is UASQ.

Let X be a Banach space. Then X^* can not be UASQ.

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Let X and Y be Banach spaces, and let $0 < \alpha < 1$. Assume that X^* is separable. Then $lip_{\omega^*}((B_{X^*}, \| \|^\alpha), Y)$ is UASQ.

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In particular, above spaces are not isometric to any dual Banach space. Previous result has an immediate consequence in terms of octahedrality. It follows from previous result that $\mathcal{F}((B_{X^*},\|\ \|^\alpha),Y^*)$ has an octahedral norm whenever X^* is separable. This answers partially a question posed by Becerra, López and Rueda, who wondered whether octahedrality in vector-valued Lipschitz-free Banach spaces actually relies on the scalar case.

References

Abrahamsen, T., J. Langemets, and V. Lima. "Almost square Banach spaces". In: *J. Math. Anal. Appl.* 434.2 (2016), pp. 1549–1565.

Dalet, A. "Free spaces over some proper metric spaces". In: *Mediterr. J. Math.* 12.3 (2015), pp. 973–986.

García-Lirola, L., C. Petitjean, and A. Rueda Zoca. "On the structure of spaces of vector-valued Lipschitz functions". arXiv. 2016. URL: arxiv.org/abs/1606.05999.

García-Lirola, L. and A. Rueda Zoca. "Unconditional almost squareness and applications to spaces of Lipschitz functions". arXiv. 2016. URL: arxiv.org/abs/1605.04699.

Godefroy, G. and N. J. Kalton. "Lipschitz-free Banach spaces". In: *Studia Math.* 159.1 (2003), pp. 121–141.

Guerrero, J. Becerra, G. López-Pérez, and A. Rueda Zoca. "Octahedrality in Lipschitz free Banach spaces". arXiv. 2015. URL: arxiv.org/abs/1512.03558.

Jiménez-Vargas, A., J. M. Sepulcre, and M. Villegas-Vallecillos. "Lipschitz compact operators". In: *J. Math. Anal. Appl.* 415.2 (2014), pp. 889–901.

Kalton, N.J. "Spaces of Lipschitz and Hölder functions and their applications". In: *Collect. Math.* 55.2 (2004), pp. 171–217.

Weaver, N. *Lipschitz algebras*. World Scientific Publishing Co., 1999, pp. xiv+223.

Thank you for your attention