Volume product and Lipschitz-free Banach spaces

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Kent State University

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Región de Murcia

GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

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Mahler's conjecture (1939)

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Results by Mahler, Saint Raymond, Gordon, Meyer, Reisner, Nazarov, Stancu, Schütt, Werner, Petrov, Ryabogin, Zvavitch, Barthe, Fradelizi, Artstein-Avidan, Karasev, Ostrover, Bourgain, Milman, Giannopoulos, Paouris, Vritsiou, Kupenberg, Iriyeh, Shibata...

Lipschitz-free spaces

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We identify $f \equiv (f(a_1), \ldots, f(a_n)) \in \mathbb{R}^n$. Then

$$B_{\mathrm{Lip}_0(M)} = \left\{ f : \frac{f(a_i) - f(a_j)}{d(a_i, a_j)} \leq 1 \; \forall i \neq j \right\} = \left\{ f : \langle f, \frac{e_i - e_j}{d(a_i, a_j)} \rangle \leq 1 \; \forall i \neq j \right\}$$

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$$\mathcal{P}(M) := \mathrm{vol}_n(B_{\mathcal{F}(M)}) \cdot \mathrm{vol}_n(B_{\mathrm{Lip}_0(M)})$$

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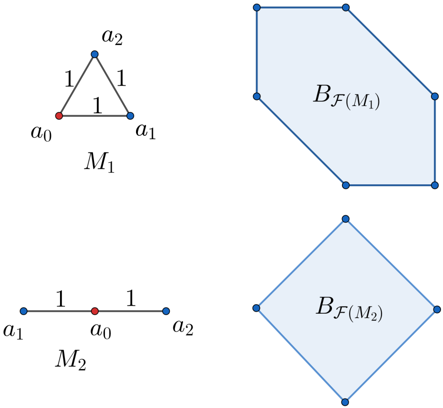

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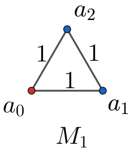
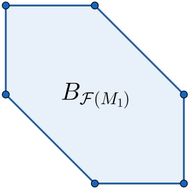
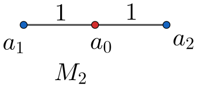
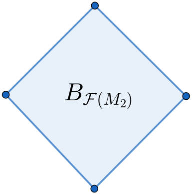
Theorem (Aliaga-Guirao, 2019)

$\frac{e_i - e_j}{d(a_i, a_j)}$ is a vertex of $B_{\mathcal{F}(M)}$ if and only if $d(x, y) < d(x, z) + d(z, y)$ for all $z \in M \setminus \{x, y\}$









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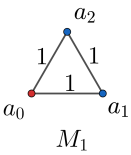

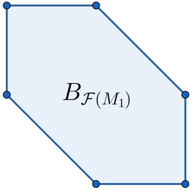

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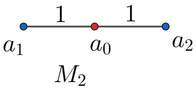

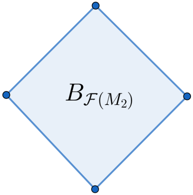

Theorem (Godard, 2010)

M is a tree if and only if $B_{\mathcal{F}(M)}$ is a linear image of B_1^n.

Minimal volume product

Let M be a finite metric space with minimal volume product such that $B_{\mathcal{F}(M)}$ is a simplicial polytope. Then M is a tree (and so $\mathcal{P}(M) = \mathcal{P}(B_1^n)$).

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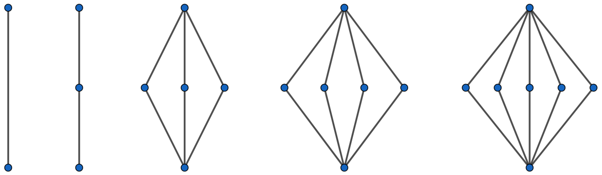
$B_{\mathcal{F}(M)}$ is a Hanner polytope if and only if M can be obtained by "joining" the following graphs:



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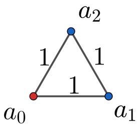
For $n = 2$, the metric space with maximum volume product is



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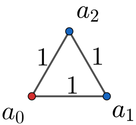

Let M be a finite metric space such that $\mathcal{P}(M)$ is maximal among the metric spaces with the same number of elements. Then

- $d(x, y) < d(x, z) + d(z, y)$ for all different points $x, y, z \in M$.
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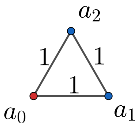

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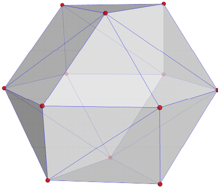
If $n \geq 3$ and M is the complete graph with equal weights, then $B_{\mathcal{F}(M)}$ is not simplicial!



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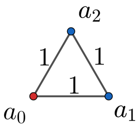

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Thank you for your attention