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Volume product and Lipschitz-free Banach spaces

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Volume product and Mahler's conjecture

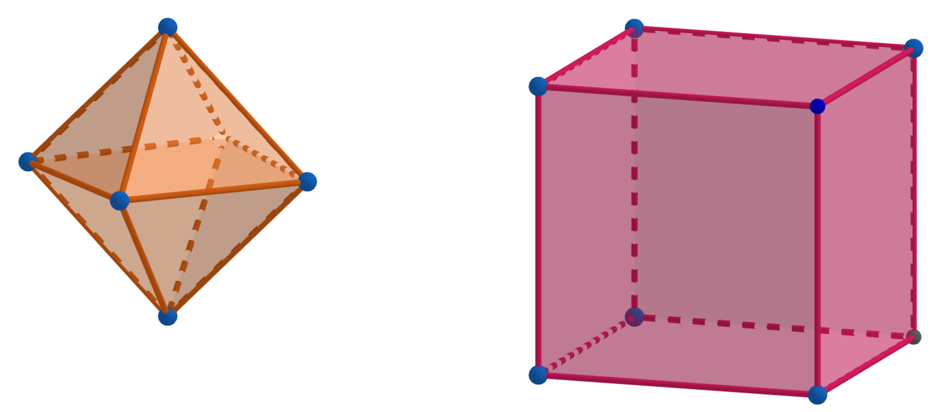
The **volume product** of an origin-symmetric convex body $K \subset \mathbb{R}^n$ is

$$\mathcal{P}(K) := \mathrm{vol}_n(K) \cdot \mathrm{vol}_n(K^\circ)$$

where K° is the *polar body* of K, i.e.

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \forall y \in K\}.$$

Note that if K is the unit ball of some norm in $\mathbb{R}^n$, then K° is the unit ball of the dual norm.



The volume product is invariant under invertible linear transformations on $\mathbb{R}^n$. It follows from the continuity of the volume function in the Banach–Mazur compactum that the volume product attains its maximum and minimum.

Theorem (Blaschke-Santaló)

Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then

$$\mathcal{P}(K) \leq \mathcal{P}(B_2^n)$$

where B_2^n is the Euclidean ball in $\mathbb{R}^n$.

Blaschke (1917) proved that inequality for $n = 2, 3$ and Santaló (1949) for general n. The equality holds if and only if K is an ellipsoid (Saint-Raymond, 1981). Modern versions of proof uses Steiner Symmetrization (Meyer-Pajor, 1990).

The minimum value of $\mathcal{P}(K)$ is an intriguing problem. Mahler (1939) proved that

$$\mathcal{P}(K) \geq \frac{4^n}{(n!)^2}$$

and applied this result to geometric number theory. He conjectured the following:

Mahler's conjecture

Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then

$$\mathcal{P}(K) \geq \mathcal{P}(B_1^n) = \mathcal{P}(B_\infty^n) = \frac{4^n}{n!}$$

The conjectured minimizers for the volume product are the **Hanner polytopes**, i.e. the unit balls of spaces which are obtaining by taking ℓ_1 or ℓ_∞ sums of ℓ_1^n or ℓ_∞^n.

Mahler's conjecture is known to be true in a number of cases:

- $n = 2$ (Mahler, 1939).
- $n = 3$. (Iriyeh-Shibata, 2017; short proof by Fradelizi-Hubard-Meyer-Roldán Pensado-Zvavitch, 2019).
- Unconditional bodies (Saint Raymond, 1981, short proof by Meyer, 1986)
- Zonoids, that is, K° is the unit ball of a subspace of L_1 (Reisner, 1986; short proof by Gordon-Meyer-Reisner, 1988).
- Bodies which are small perturbations of B_∞^n (Nazarov-Petrov-Ryabogin-Zvavitch, 2010) and of Hanner polytopes (Kim, 2014).
- Bodies having hyperplane symmetries which fix only one common point (Barthe-Fradelizi, 2013).
- Hyperplane sections of ℓ_p-balls and Hanner polytopes (Karasev, 2019).
- It is known that bodies with some positive curvature assumption (Stancu, 2009; Reisner-Schütt-Werner, 2012) are not local minimizers for the volume product.

An isomorphic version of the conjectures was proved by Bourgain and Milman (1987): there is a universal constant $c > 0$ such that $\mathcal{P}(K) \geq c^n \mathcal{P}(B_2^n)$ (other proofs by Nazarov, 2012; and Giannopoulos-Paouris-Vritsiou, 2014). The best known result in arbitrary dimension is $\mathcal{P}(K) \geq \frac{\pi^n}{n!}$ (Kuperberg, 2008).

Lipschitz-free spaces and spaces of Lipschitz functions

Given a metric space (M, d) with a distinguished point a_0, we consider the space $\mathrm{Lip}_0(M)$ of Lipschitz functions on M vanishing at a_0. $\mathrm{Lip}_0(M)$ is a dual Banach space when it is endowed with the norm given by the Lipschitz constant. Its canonical predual is the *Lipschitz-free space* over M,

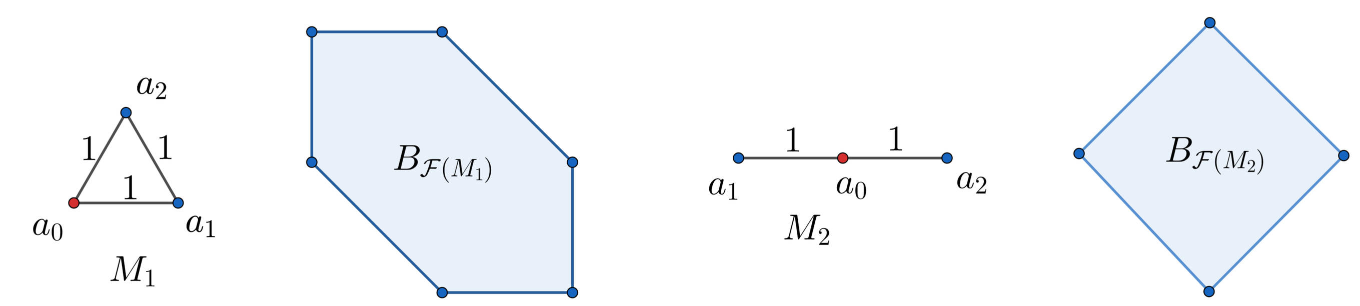
$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M)^*,$$

where $\langle \delta(x), f\rangle = f(x)$ for $f \in \mathrm{Lip}_0(M)$.

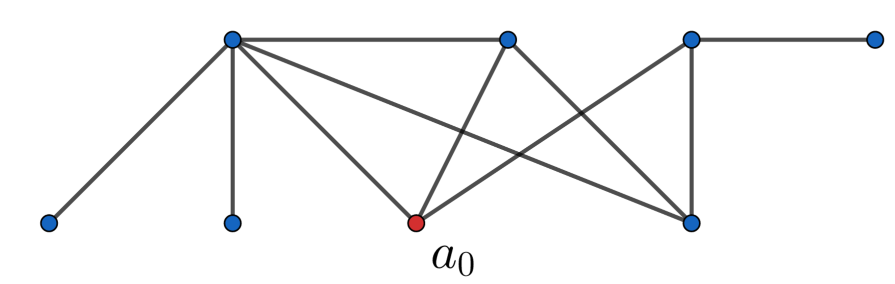
Lipschitz-free spaces have become a very active research topic. They have a number of applications on Non-Linear Analysis, for infinite metric spaces, and on Computer Science and Optimal Transportation, in the finite case.

Here we focus on the case in which $M = \{a_0, a_1, \dots, a_n\}$ is finite. Then we can identify each function $f \in \mathrm{Lip}_0(M)$ with the vector $(f(a_1), \dots, f(a_n)) \in \mathbb{R}^n$. So we can identify $B_{\mathrm{Lip}_0(M)}$ and $B_{\mathcal{F}(M)}$ with certain convex bodies in $\mathbb{R}^n$, indeed,

$$B_{\mathcal{F}(M)} = \mathrm{conv}\left\{\frac{e_i - e_j}{d(a_i, a_j)} : i \neq j\right\}.$$



Every finite metric space M is represented by a weighted graph, where the nodes are the points of M and the weight of the edges are the distances between them. We agree that (x, y) is an edge of the graph if and only if $d(x, y) < d(x, z) + d(z, y)$ for all $z \in M \setminus \{x, y\}$.



This representation is very well adapted to our study. Indeed,

- The point $\frac{e_i - e_j}{d(a_i, a_j)}$ is an extreme point of $B_{\mathcal{F}(M)}$ if and only if
$$d(x, y) < d(x, z) + d(z, y) \text{ for all } z \in M \setminus \{x, y\}$$
(Aliaga-Guirao, 2019). Thus, in the finite setting, the vertices of the polytope $B_{\mathcal{F}(M)}$ correspond to the edges in the canonical graph of M.
- $\mathcal{F}(M)$ is isometric to ℓ_1^n if and only if $n = 3$ and M is a tree (Godard, 2010).

Besides, one can check that $\mathcal{F}(M)$ is isometric to ℓ_∞^n with $n \geq 3$ if and only if $n = 3$ and M is a regular cycle with 4 points.

The volume product of a metric space

We define

$$\mathcal{P}(M) := \mathcal{P}(B_{\mathcal{F}(M)}) = \mathrm{vol}_n(B_{\mathcal{F}(M)})\,\mathrm{vol}_n(B_{\mathrm{Lip}_0(M)}).$$

It follows from the properties of Lipschitz-free spaces that $\mathcal{P}(M)$ is an isometric invariant of M. Our goal is to study the maximum and the minimum values of the volume product of M.

Is it true that $\mathcal{P}(M) \geq \frac{4^n}{n!}$ for every metric space with $n + 1$ elements?

Note also that $B_{\mathcal{F}(M)}$ is a polytope with at most $n(n+1)$ vertices. Thus, it also makes sense to wonder about the maximum value for $\mathcal{P}(M)$.

For which metric spaces M is $\mathcal{P}(M)$ maximal among the metric spaces with the same number of elements?

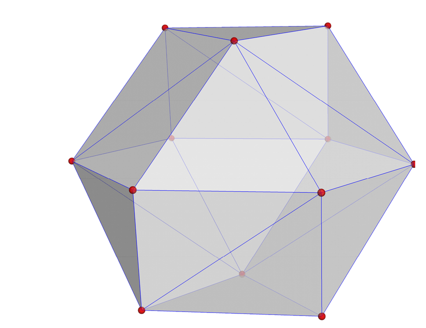
The maximum value of $\mathcal{P}(M)$

One can check that the maximum of $\mathcal{P}(M)$ among metric spaces with 3 elements is attained at the metric space corresponding to the complete graph with equal weights. To deal with the maximum in general dimension we use the shadow-system technique and ideas coming from Alexander-Fradelizi-Zvavitch (2019).

Theorem A

Let M be a finite metric space such that $\mathcal{P}(M)$ is maximal among the metric spaces with the same number of elements. Then $B_{\mathcal{F}(M)}$ is a simplicial polytope and $d(x, y) < d(x, z) + d(z, y)$ for all different points $x, y, z \in M$.

It is easy to check that $B_{\mathcal{F}(M)}$ is not simplicial in the case corresponding to the complete graph with equal weights, provided the metric space has at least 4 points. Thus, the volume product of that metric space is not maximum.



The minimum value of $\mathcal{P}(M)$

It is already known that $\mathcal{P}(M) \geq \frac{4^n}{n!}$ for certain metric spaces. Namely:

- If M embeds into a tree, then $B_{\mathrm{Lip}_0(M)}$ is a zonoid (Godard, 2010).
- If M is a cycle, then $B_{\mathcal{F}(M)}$ has $2n + 2$ vertices and so it is a section of B_∞^{n+1}.

By using shadow systems and the structure of $B_{\mathcal{F}(M)}$, we get the following.

Theorem B

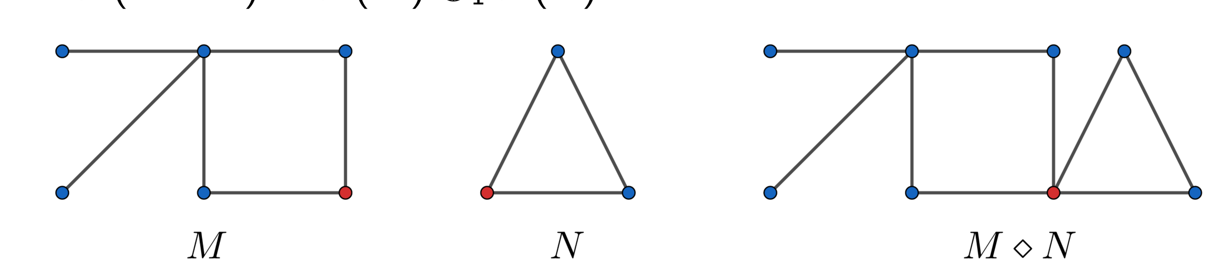
Let M be a finite metric space with minimal volume product such that $B_{\mathcal{F}(M)}$ is a simplicial polytope. Then M is a tree (and so $\mathcal{F}(M)$ is isometric to ℓ_1^n).

The minimal case for four points corresponds to the question on the minimality of volume product in $\mathbb{R}^3$. That question was solved by Iriyeh-Shibata (2017) and automatically gives $\mathcal{P}(M) \geq \mathcal{P}(B_1^3)$, for M being a metric space of four elements. Our techniques also lead to a direct proof.

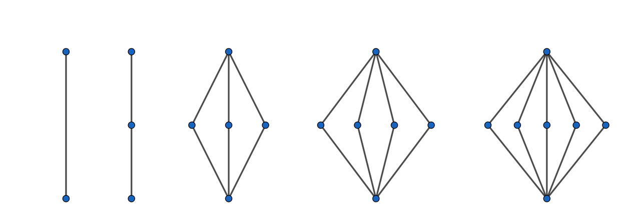
Corollary

Let M be a metric space with four points. Then $\mathcal{P}(M) \geq \mathcal{P}(B_1^3)$. Equality holds if and only if M is a tree or a regular cycle.

We also characterize the metric spaces such that $B_{\mathcal{F}(M)}$ is a Hanner polytope. To this end, we introduce some notation. The ℓ_1*-sum* of two finite metric spaces M, N is the metric space $M \diamond N$ obtained by identifying the distinguished points of M and N. Note that $\mathcal{F}(M \diamond N) = \mathcal{F}(M) \oplus_1 \mathcal{F}(N)$.



A metric space is a *spiderweb* if it contains only two points or it is the complete bipartite graph $K_{2,n}$, where all the edges have the same weight.



Theorem C

$B_{\mathcal{F}(M)}$ is a Hanner polytope if and only if $M = M_1 \diamond \ldots \diamond M_r$ and each M_i is a spiderweb.

To prove that theorem, we characterize the finite metric spaces M such that $\mathcal{F}(M)$ admits a non-trivial decomposition as $X \oplus_1 Y$ or $X \oplus_\infty Y$.

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