Stability of Lipschitz-type functions under pointwise product and reciprocation

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Joint work with Gerald Beer and Maribel Garrido

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March 10th, 2020

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GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

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- If $f^2 \in V$ whenever $f \in V$, then V is closed under pointwise products.
- Let V be a lattice containing the constants. If V is closed under reciprocation, then V is closed under pointwise products.

1. Lipschitz functions
2. Locally Lipschitz functions
3. Cauchy-Lipschitz functions
4. Uniformly locally Lipschitz functions
5. Lipschitz in the small functions

Lipschitz functions

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$$V = \{f : X \to \mathbb{R} : f|_A \text{ is Lipschitz } \forall A \in \mathcal{A}\}$$

- V is closed under pointwise product if and only if every element in $\mathcal{A}$ is bounded.
- V is closed under reciprocation if and only if every element in $\mathcal{A}$ is relatively compact.

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Theorem (Beer-G.-Garrido)

- CL(X) *is always closed under pointwise product.*
- *The following are equivalent:*
  - *(a)* CL(X) *is closed under reciprocation.*
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(b)$\Leftrightarrow$(c) was shown by (Beer-Garrido, 2016).

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- $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is uniformly locally Lipschitz but not Lipschitz.
- $X = \{kx_0 + \frac{1}{k}e_n, k, n \in \mathbb{N}\}$, $f : X \to \mathbb{R}$ given by $f(kx + \frac{1}{k}e_n) = n$. Then f is Cauchy-Lipschitz but not uniformly locally Lipschitz.

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- ULL(X) *is always closed under pointwise product.*
- *The following are equivalent:*
  (a) ULL(X) *is closed under reciprocation.*
  (b) Every locally Lipschitz function on X is uniformly locally Lipschitz.
  (c) X is cofinally complete.

(b)⇔(c) was shown by (Beer-Garrido, 2015).

Uniformly locally Lipschitz functions

Definition (Beer, 2008)

$\{x_n\}$ is *cofinally Cauchy* if for all $\varepsilon > 0$ there is an infinite subset $\mathbb{N}_\varepsilon \subset \mathbb{N}$ such that for all $i, j \in \mathbb{N}_\varepsilon$ we have $d(x_i, x_j) < \varepsilon$.

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- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is uniformly locally Lispchitz but not Lipschitz in the small.
- $X = \bigcup_{n=1}^{\infty} [n - \frac{1}{4}, n + \frac{1}{4}] \subset \mathbb{R}$, $f : X \to \mathbb{R}$ defined by $f(x) = n^2$ if $n - \frac{1}{4} \leq x \leq n + \frac{1}{4}$ is Lipschitz in the small but fails to be Lipschitz.

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Theorem (Garrido-Jaramillo, 2008)

Every uniformly continuous function can be uniformly approximated by Lipschitz in the small functions.

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Theorem (Garrido-Jaramillo, 2008)

Every uniformly continuous function can be uniformly approximated by Lipschitz in the small functions.

Theorem (Cabello-Sánchez, 2017)

The space of uniformly continuous functions on X is stable under pointwise product if and only if every subset of X is either Bourbaki bounded or contains an infinite uniformly discrete subset.

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- LS(X) *is closed under pointwise product if and only if every subset of* X *is either Bourbaki bounded or contains an infinite uniformly discrete subset.*

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- $\mathrm{LS}(X)$ *is closed under pointwise product if and only if every subset of* X *is either Bourbaki bounded or contains an infinite uniformly discrete subset.*
- *The following are equivalent:*
  (a) $\mathrm{LS}(X)$ *is closed under reciprocation.*
  (b) Every locally Lipschitz function is Lipschitz in the small.
  (c) X *is a UC-space (i.e. every continuous function on* X *is uniformly continuous).*

(b)$\Leftrightarrow$(c) was shown by (Beer-Garrido, 2015).

Thank you for your attention!