Norm-attainment in projective tensor products

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Joint works with S. Dantas, J. Guerrero-Viu, M. Jung, A. Rueda Zoca

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GOBIERNO DE ESPAÑA
MINISTERIO DE CIENCIA E INNOVACIÓN

- S. Dantas, M. Jung, O. Roldán, A. Rueda Zoca. Norm-attaining tensors and nuclear operators. *Mediterr. J. Math.* 19 (2022), no. 1, Paper No. 38, 27 pp.
- S. Dantas, L. C. García-Lirola, M. Jung, A. Rueda Zoca. On norm-attainment in (symmetric) tensor products. *Quaest. Math.* 46 (2023), no. 2, 393–409.
- L. C. García-Lirola, J. Guerrero-Viu, A. Rueda Zoca. Projective tensor products where every element is norm-attaining. arXiv:2407.10710

Outline

- Tensor products
- Projective tensor products where every tensor attains its norm
- Denseness of norm-attaining tensors

Tensor products

$X, Y, Z =$ vector spaces

A map $A\colon X \times Y \to Z$ is **bilinear** if

- $A(\alpha x + \beta x', y) = \alpha A(x, y) + \beta A(x', y)$
- $A(x, \alpha y + \beta y') = \alpha A(x, y) + \beta A(x, y')$

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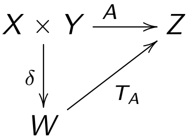
We would like to **linearize** bilinear maps. That is, to find a vector space W and a linear embedding $\delta\colon X \times Y \to W$ such that for any $A \in B(X \times Y, Z)$ there is a linear map $T_A\colon W \to Z$ such that this diagram commutes:



Tensor products

Given $x \in X$ and $y \in Y$, consider the linear functional

$$\begin{aligned} x \otimes y \colon B(X \times Y, \mathbb{R}) &\to \mathbb{R} \\ A &\mapsto A(x, y) \end{aligned}$$

The **tensor product** $X \otimes Y$ is defined as

$$X \otimes Y = \operatorname{span}\{x \otimes y : x \in X,\ y \in Y\} \subset B(X \times Y, \mathbb{R})^{\#}$$

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A **tensor** $u \in X \otimes Y$ has the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in X$ and $y_i \in Y$. This representation is **NOT UNIQUE**.
Indeed,

- $(\alpha x + \beta x') \otimes y = \alpha x \otimes y + \beta x' \otimes y$.
- $x \otimes (\alpha y + \beta y') = \alpha x \otimes y + \beta x \otimes y'$

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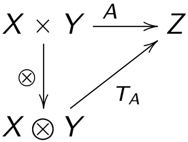
Define $\otimes\colon X \times Y \to X \otimes Y$ by $\otimes(x, y) = x \otimes y$.



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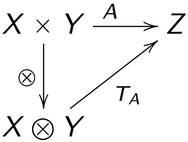

Given $A \in B(X \times Y, Z)$, define

$$T_A\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} A(x_i, y_i).$$

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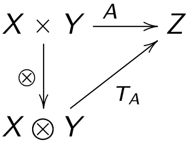

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Conversely, if $T\colon X \otimes Y \to Z$ is linear, define $A \in B(X \times Y, Z)$ by

$$A(x, y) = T(x \otimes y).$$

The projective norm

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Given $u \in X \otimes Y$, we define the **projective norm** as follows

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$L_1(\mu) \hat{\otimes}_\pi X = L_1(\mu, X)$

The dual of $X\widehat{\otimes}_\pi Y$

A bilinear map $A\colon X \times Y \to Z$ is said to be **bounded** if there is $C > 0$ such that

$$\|A(x,y)\| \leqslant C\|x\|\|y\|, \quad \forall x \in X, y \in Y.$$

$\mathcal{B}(X \times Y, Z)$ denotes the Banach space of bounded bilinear mappings with the norm $\|A\| = \sup\{\|A(x,y)\| : \|x\| \leqslant 1, \|y\| \leqslant 1\}$.

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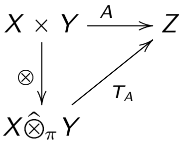

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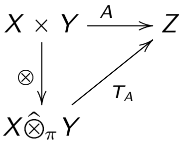

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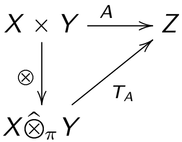

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Since $\|A\| = \sup\{\langle A, x \otimes y\rangle : x \in B_X, y \in B_Y\}$, it follows that $B_{X\widehat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y)$.

Norm-attainment

Every $u \in X \widehat{\otimes}_\pi Y$ can be written as $u = \sum_{i=1}^{\infty} x_i \otimes y_i$ with $x_i \in X$, $y_i \in Y$.

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Dantas-Jung-Roldán-Rueda Zoca (2022)

We say that $u \in X \widehat{\otimes}_\pi Y$ **attain its projective norm** if

$$u = \sum_{i=1}^{\infty} x_i \otimes y_i \quad \text{and} \quad \|u\|_\pi = \sum_{i=1}^{\infty} \|x_i\| \|y_i\|.$$

The set of those u is denoted by $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

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Most of the results can be translated to other settings:

- Nuclear operators (tensors in $X^*\widehat{\otimes}_\pi Y$ correspond to operators $X \to Y$)
- Symmetric tensors ($\widehat{\otimes}_{\pi,s,N} X$ is a predual of $\mathcal{P}(^N X)$)

When every element attains its projective norm?

Dantas, Jung, Roldán, Rueda Zoca (2022)

$\mathrm{NA}(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ in the following cases:

a) X and Y are finite dimensional.

b) $X = \ell_1(I)$ and Y is any Banach space.

c) X and Y are complex Hilbert spaces.

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Sketch of the proof.

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c) Diagonalization

When every tensor attains its projective norm?

Example (Dantas, Jung, Roldán, Rueda Zoca (2022))

Consider $u = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \otimes e_n \in \ell_1 \widehat{\otimes}_\pi \ell_2$.

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Indeed, a representation $u = \sum_{n=1}^{\infty} x_n \otimes y_n \in X \widehat{\otimes}_\pi Y$ is optimal if and only if there is $T : X \to Y^*$ such that $T(x_n)(y_n) = \|x_n\| \, \|y_n\| \ \forall n$

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Dantas, Jung, Roldán, Rueda Zoca (2022)

If every tensor on $X \widehat{\otimes}_\pi Y$ attains its projective norm, then every operator $T\colon X \to Y^*$ can be approximated by norm-attaining ones.

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If every tensor on $X \widehat{\otimes}_\pi Y$ attains its projective norm, then every operator $T\colon X \to Y^*$ can be approximated by norm-attaining ones.

Thus, there are tensors not attaining its projective norm in $L_1(\mathbb{T}) \widehat{\otimes}_\pi \ell_2^2$, $L_1[0,1] \widehat{\otimes}_\pi L_1[0,1]$, $\ell_p \widehat{\otimes}_\pi G$ $(1 < p < +\infty)$,...

When every tensor attains its projective norm?

Dantas, G.L., Jung, Rueda Zoca (2023)

Let X be a finite-dimensional polyhedral Banach space and Y is any dual Banach space. Then $\mathrm{NA}_{\pi}(X\widehat{\otimes}_{\pi}Y) = X\widehat{\otimes}_{\pi}Y$

When every tensor attains its projective norm?

Dantas, G.L., Jung, Rueda Zoca (2023)

Let X be a finite-dimensional polyhedral Banach space and Y is any dual Banach space. Then $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$

Sketch of the proof.

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$$\begin{array}{ccc} X & \longrightarrow & \ell_\infty^n \\ x & \longmapsto & (x_1^*(x), \ldots, x_n^*(x)) \end{array}$$

is a linear into isometry since

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Therefore, X is a subspace of Z with $Z^* = \ell_1^n$.

G.L., Guerrero-Viu, Rueda Zoca

Let Z be such that $Z^* = \ell_1(I)$, $X \subset Z$ and Y be any dual space. If either X^* or Y has the approximation property, then $\text{NA}_\pi(X^* \widehat{\otimes}_\pi Y) = X^* \widehat{\otimes}_\pi Y$.

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We can take X^* the **Lipschitz-free space** $\mathcal{F}(M)$ for a metric space M satisfying one of the following conditions:

a) M is countable and compact.

b) M is uniformly discrete, countable, and there is a compact Hausdorff topology τ on M such that d is τ-lower semicontinuous, and $V = \{d(x,y) \, : \, (x,y) \in M^2\} \subseteq \mathbb{R}_0^+$ is a compact set.

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Lemma

Let X, Y be normed spaces and $T\colon X \to Y$ a linear isometry (into). Then, the operator $T^*\colon Y^* \to X^*$ is a quotient operator. In fact, given $x^* \in X^*$, there exists $y^* \in Y^*$ such that $T^*(y^*) = x^*$ and $\|y^*\| = \|x^*\|$.

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Sketch of the proof of the theorem.

Say $Y = E^*$. Then $T \colon X \widehat{\otimes}_\varepsilon E \to Z \widehat{\otimes}_\varepsilon E$ given by $T(u) = u$ is a linear isometry.

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Given $u \in X^* \widehat{\otimes}_\pi Y$ there is $\tilde{u} \in Z^* \widehat{\otimes}_\pi Y = \ell_1(I) \widehat{\otimes}_\pi Y$ with $T^*(\tilde{u}) = u$ and the same norm.

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$\tilde{u}$ admits an optimal representation $\Rightarrow$ the same holds for u.

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Indeed, we have shown:

G.L., Guerrero-Viu, Rueda Zoca

Let Z be an Asplund space, $X \subset Z$ and Y be any dual space such that $\mathrm{NA}_\pi(Z^* \widehat{\otimes}_\pi Y) = Z^* \widehat{\otimes}_\pi Y$. If either X^* or Y has the approximation property, then $\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y) = X^* \widehat{\otimes}_\pi Y$.

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Also, in the above results the hypothesis of Y being a dual space can be weakened to Y being 1-complemented in its bidual.

G.L., Guerrero-Viu, Rueda Zoca

Let X, Y be Banach spaces such that $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$. Assume $Z \subset Y$ is a 1-complemented subspace. Then, $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Z) = X \widehat{\otimes}_\pi Z$.

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Recall that every tensor in $\ell_1(I)\widehat{\otimes}_\pi Y$ attains its projective norm for any Y.

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Assume that X is separable and $\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y) = X^*\widehat{\otimes}_\pi Y$ for any Y.
Then $B_X = \overline{\mathrm{conv}}(\exp B_X)$.

Denseness of norm-attaining tensors

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There exist X, Y such that $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$ is **NOT** dense in $X \widehat{\otimes}_\pi Y$.

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A space X is said to have the **metric π-property** if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset S_X$, we can find a finite-dimensional 1-complemented subspace $M \subset X$ and points $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon \ \forall i$.

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Equivalently, there are finite-rank projections $P_\alpha \colon X \to X$ with $\|P_\alpha\| = 1$ and $P_\alpha x \to x$ for every $x \in X$.

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Key point: If M is 1-complemented in X then $M\widehat{\otimes}_\pi Y$ is (isometrically) a subspace of $X\widehat{\otimes}_\pi Y$.

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$\mathrm{NA}_\pi(c_0 \widehat{\otimes}_\pi Y)$ is dense in $c_0 \widehat{\otimes}_\pi Y$ for any dual space Y.

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More generally, $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense if X is a polyhedral space with the metric π-property and Y is any dual space.

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Suppose that for every $\varepsilon > 0$ and $x_1, \ldots, x_n \in X$, there exists a finite dimensional 1-complemented subspace $M \subseteq X$ and $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$ for each $i = 1, \ldots, n$. Assume that $\overline{\mathrm{NA}_\pi(M\hat{\otimes}_\pi Y)} = M\hat{\otimes}_\pi Y$. Then,

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$\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$ is dense in $X \widehat{\otimes}_\pi Y$ in the following cases:

a) $X = L_1(\mu)$ for some measure μ, and Y is any Banach space.

b) $X^* = L_1(\mu)$ and Y is 1-complemented in Y^{**}.

c) X has the metric π-property and Y is a dual space with the RNP.

Denseness of norm-attaining tensors

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Let X, Y be reflexive Banach spaces. Is $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ dense in $X\widehat{\otimes}_\pi Y$?

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Given $u \in B_{X\hat{\otimes}_\pi Y}$ with $\|u\|_\pi = 1$, take $A \in (X\hat{\otimes}_\pi Y)^*$ with $\langle A, u\rangle = 1$.

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$F = \overline{\mathrm{conv}}(\mathrm{ext}\, F)$. Since F is a face of $B_{X\hat{\otimes}_\pi Y}$, $\mathrm{ext}\, F \subset \mathrm{ext}\, B_{X\hat{\otimes}_\pi Y}$.

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$F = \overline{\mathrm{conv}}(\mathrm{ext}\, F)$. Since F is a face of $B_{X\hat{\otimes}_\pi Y}$, $\mathrm{ext}\, F \subset \mathrm{ext}\, B_{X\hat{\otimes}_\pi Y}$.
Under these hypotheses, $\mathrm{ext}(B_{X\hat{\otimes}_\pi Y}) \subset B_X \otimes B_Y$ (Collins-Ruess (1983))

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Assume X and Y are dual spaces with the RNP and at least one of them has the approximation property. Then $\mathrm{NA}_\pi(X\hat{\otimes}_\pi Y)$ dense in $X\hat{\otimes}_\pi Y$.

Sketch of the proof

Given $u \in B_{X\hat{\otimes}_\pi Y}$ with $\|u\|_\pi = 1$, take $A \in (X\hat{\otimes}_\pi Y)^*$ with $\langle A, u\rangle = 1$.
Consider $F = \{z \in B_{X\hat{\otimes}_\pi Y} : \langle A, z\rangle = 1\}$.
$F = \overline{\mathrm{conv}}(\mathrm{ext}\, F)$. Since F is a face of $B_{X\hat{\otimes}_\pi Y}$, $\mathrm{ext}\, F \subset \mathrm{ext}\, B_{X\hat{\otimes}_\pi Y}$.
Under these hypotheses, $\mathrm{ext}(B_{X\hat{\otimes}_\pi Y}) \subset B_X \otimes B_Y$ (Collins-Ruess (1983))
Thus,

$$u \in \overline{\mathrm{conv}}\{x \otimes y \in S_X \otimes S_Y : A(x, y) = 1\}$$

Some related questions

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Are there X, Y such that the set of operators attaining its nuclear norm is **NOT** dense in $\mathcal{N}(X, Y)$?

Is there X such that the set of symmetric tensors attaining its projective norm is **NOT** dense in $\widehat{\otimes}_{\pi,s,N} X$?

Thank you for your attention!