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Laplacian Flow of Closed G_2-Structures Inducing Nilsolitons

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Abstract We study the existence of left invariant closed G_2-structures defining a Ricci soliton metric on simply connected nonabelian nilpotent Lie groups. For each one of these G_2-structures, we show long time existence and uniqueness of solution for the Laplacian flow on the noncompact manifold. Moreover, considering the Laplacian flow on the associated Lie algebra as a bracket flow on $\mathbb{R}^7$ in a similar way as in Lauret (Commun Anal Geom 19(5):831–854, 2011) we prove that the underlying metrics $g(t)$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in the nilpotent Lie group, as t goes to infinity.

Keywords Closed G_2-structure · Nilsoliton · Laplacian flow

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1 Introduction

A G_2-structure on a 7-dimensional manifold M can be characterized by the existence of a globally defined 3-form φ, which is called the G_2 form or the fundamental 3-form and it can be described locally as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some local basis $\{e^1, \dots, e^7\}$ of the 1-forms on M.

There are many different G_2-structures attending to the behavior of the exterior derivative of the G_2 form [2,13]. In the following, we will focus our attention on *closed G_2-structures* which are characterized by the closure of the G_2 form.

The existence of a G_2 form φ on a manifold M induces a Riemannian metric g_φ on M given by

$$g_\varphi(X, Y) vol = \frac{1}{6}\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi, \tag{1}$$

for any vector fields X, Y on M, where vol is the volume form on M.

By [3,6] a closed G_2-structure on a compact manifold cannot induce an Einstein metric, unless the induced metric has holonomy contained in G_2. It is still an open problem to see if the same property holds on noncompact manifolds. For the homogeneous case, a negative answer has been recently given in [12]. Indeed, we showed that if a solvable Lie algebra has a closed G_2-structure then the induced inner product is Einstein if and only if it is flat.

Natural generalizations of Einstein metrics are given by Ricci solitons, which have been introduced by Hamilton in [14]. A natural question is thus to see if a closed G_2-structure on a noncompact manifold induces a (non-Einstein) Ricci soliton metric. In this paper we give a positive answer to this question, showing that there exist 7-dimensional simply connected nonabelian nilpotent Lie groups with a closed G_2-structure which determines a left invariant Ricci soliton metric.

All known examples of nontrivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups, whose Ricci operator satisfies the condition

$$Ric(g) = \lambda I + D,$$

for some $\lambda \in \mathbb{R}$ and some derivation D of the corresponding Lie algebra. The left invariant metrics satisfying the previous condition are called *nilsolitons* if the Lie groups are nilpotent [19]. Not all nilpotent Lie groups admit nilsoliton metrics, but if a nilsoliton exists, then it is unique up to automorphism and scaling [19]. The nilsolitons metrics are strictly related to left invariant Einstein metrics on solvable Lie groups. Indeed, by [20], a simply connected nilpotent Lie group N admits a nilsoliton metric if and only if its Lie algebra $\mathfrak{n}$ is an Einstein nilradical, which means that $\mathfrak{n}$ has an inner product $\langle\cdot,\cdot\rangle$ such that there is a metric solvable extension of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ which is Einstein. According to [15,21], such an Einstein metric has to be of standard type and it is unique, up to isometry and scaling.

Seven dimensional nilpotent Lie algebras admitting a closed G_2-structure have been recently classified in [8], showing that there are twelve isomorphism classes, including the abelian case which has a trivial nilsoliton because it is flat. A classification of 7-dimensional nilpotent Lie algebras admitting a nilsoliton has been recently given in [11], but the explicit expression of the nilsoliton is not written in all the cases that we need.

Using the classification in [8] and Table 1 in [9], we have that, up to isomorphism, there is a unique nilpotent Lie algebra with a closed G_2 form but not admitting nilsolitons. It turns out that all the other ten nilpotent Lie algebras have a nilsoliton, and we can determine explicitly the nilsoliton except for the Lie algebra $\mathfrak{n}_{10}$ which is 4-step nilpotent (see also [9–11]). In Proposition 3.4 we prove that the Lie algebra $\mathfrak{n}_i$ ($i = 3, 5, 7, 8, 11$) has a nilsoliton but no closed G_2-structure inducing the nilsoliton. Moreover, as we mentioned before, the existence of a nilsoliton on the Lie algebra $\mathfrak{n}_{10}$ was shown in [9, Example 2], but we cannot explicit its nilsoliton. Therefore, it remains open the question of whether the Lie algebra $\mathfrak{n}_{10}$ admits a closed G_2 form inducing a nilsoliton or not. This is the reason why the result of Theorem 3.6 is restricted to s-step nilpotent Lie algebras, with $s = 2, 3$. In fact, in Theorem 3.6, we show that, up to isomorphism, there are exactly four s-step nilpotent Lie algebras ($s = 2, 3$) with a closed G_2 form defining a nilsoliton.

The Ricci flow became a very important issue in Riemannian geometry and has been deeply studied. The same techniques are also useful in the study of the flow involving other geometrical structures, like for example, the Kähler Ricci flow that was studied by Cao in [5].

For any closed G_2-structure on a manifold M, in [3] Bryant introduced a natural flow, the so-called *Laplacian flow*, given by

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t \varphi(t), \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\varphi(t)$ is a closed G_2 form on M, and Δ_t is the Hodge Laplacian operator of the metric determined by $\varphi(t)$. If the initial 3-form φ_0 is closed, then a solution $\varphi(t)$ of the flow remains closed, and the de Rham cohomology class $[\varphi(t)]$ is constant in t. The short time existence and uniqueness of solution for the Laplacian flow of any closed G_2-structure, on a compact manifold M, has been proved by Bryant and Xu in the unpublished paper [4]. Also, long time existence and convergence of the Laplacian flow starting near a torsion-free G_2-structure was proved in the unpublished paper [29].

In Sect. 4 (Theorems 4.2, 4.7, 4.8 and 4.10) we show long time existence of the solution for the Laplacian flow on the four nilpotent Lie groups admitting an invariant closed G_2-structure which determines the nilsoliton (see Theorem 3.6).

To our knowledge, these are the first examples of noncompact manifolds having a closed G_2-structure with long time existence of solution.

Since the Laplacian flow is invariant by diffeomorphisms and the initial G_2-form φ_0 is invariant, the solution $\varphi(t)$ of the Laplacian flow has to be also invariant. Therefore, we show that the Laplacian flow is equivalent to a system of ordinary differential equations which admits a unique solution. We prove that the solution for the four manifolds is defined for any $t \in [0, +\infty)$. Moreover, considering the Laplacian flow on the associated Lie algebra as a bracket flow on $\mathbb{R}^7$, in a similar way as Lauret did in [23] for the Ricci flow, we show that the underlying metrics $g(t)$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in the nilpotent Lie group as t goes to infinity. Indeed, by [23, Proposition 2.1] the convergence of the metrics in the $\mathcal{C}^\infty$ uniformly on compact sets in $\mathbb{R}^7$ is equivalent to the convergence of the nilpotent Lie brackets μ_t in the algebraic subset of nilpotent Lie brackets $\mathcal{N} \subset (\Lambda^2\mathbb{R}^7)^* \otimes \mathbb{R}^7$ with the usual vector space topology.

2 Preliminaries on Nilsolitons

In this section, we recall some definitions and results about *nontrivial homogeneous Ricci soliton metrics* and, in particular, on *nilsolitons*. For more details, see for instance [7,17,19].

A complete Riemannian metric g on a manifold M is said to be a *Ricci soliton* if its Ricci curvature tensor $Ric(g)$ satisfies the following condition

$$Ric(g) = \lambda g + \mathcal{L}_X g,$$

for some real constant λ and a complete vector field X on M, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. If in addition X is the gradient vector field of a smooth function $f : M \to \mathbb{R}$, then the Ricci soliton is said to be of *gradient type*. Ricci solitons are called *expanding*, *steady* or *shrinking* depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

In the next section we will focus our attention on nilsolitons, that is, a particular type of nontrivial homogeneous Ricci soliton metrics.

A Ricci soliton metric g on M is called *trivial* if g is an Einstein metric or g is the product of a homogeneous Einstein metric with the Euclidean metric; and g is said to be *homogeneous* if its isometry group acts transitively on M, and hence g has bounded curvature [22].

In order to characterize the nontrivial homogeneous Ricci soliton metrics, we note that any homogeneous steady or shrinking Ricci soliton metric g of gradient type is trivial. Indeed, if g is steady, one can check that g is Ricci flat, and so by [1] g must be flat. If g is shrinking, then by the results in [25, Theorem 1.2] and in [27], (M, g) is isometric to a quotient of $P \times \mathbb{R}^k$, where P is some homogeneous Einstein manifold with positive scalar curvature. Now, we should notice that this last result for shrinking homogeneous Ricci soliton metrics is also true for homogeneous Ricci solitons of gradient type [27]. Moreover, if a homogeneous Ricci soliton g on a manifold M is expanding, then by [16] M must be noncompact; and from [26] all Ricci solitons (homogeneous or nonhomogeneous) on a compact manifold are of gradient type. Therefore, as it was noticed by Lauret in [22] we have the following

Lemma 2.1 ([22]) *Let g be a nontrivial homogeneous Ricci soliton on a manifold M. Then, g is expanding and it cannot be of gradient type. Moreover, M is noncompact.*

All known examples of nontrivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups whose Ricci operator is a multiple of the identity modulo derivations, and they are called *solsolitons* or, in the nilpotent case, *nilsolitons*.

Let N be a simply connected nilpotent Lie group, and denote by $\mathfrak{n}$ its Lie algebra. A left invariant metric g on N is called a *Ricci nilsoliton metric* (or simply *nilsoliton metric*) if its Ricci endomorphism $Ric(g)$ differs from a derivation D of $\mathfrak{n}$ by a scalar multiple of the identity map I, i.e., if there exists a real number λ such that

$$Ric(g) = \lambda I + D. \tag{2}$$

Clearly, any left invariant metric which satisfies (2) is automatically a Ricci soliton.

Nilsoliton metrics have properties that make them preferred left invariant metrics on nilpotent Lie groups in the absence of Einstein metrics. Indeed, nonabelian nilpotent Lie groups do not admit left invariant Einstein metrics ([24]).

From now on, we will always identify a left invariant metric on a Lie group N with an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$ on the Lie algebra $\mathfrak{n}$ of N. A Lie algebra $\mathfrak{n}$ endowed with an inner product is usually called in the literature a *metric Lie algebra* and is denoted as the pair $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$.

We will say that a metric nilpotent Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$ is a *nilsoliton* if there exists a real number λ and a derivation D of $\mathfrak{n}$ such that

$$Ric(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}}) = \lambda I + D. \tag{3}$$

Not all nilpotent Lie algebras admit nilsoliton inner products, but if a nilsoliton inner product exists, then it is unique up to automorphism and scaling [19]. A computational method for classifying nilpotent Lie algebras having a nilsoliton inner product in a large subclass of the set of all nilpotent Lie algebras, has been recently introduced in [18]. By Lauret's results it turns out that nilsoliton metrics on simply connected nilpotent Lie groups N are strictly related to Einstein metrics on the so-called solvable rank-one extensions of N.

Definition 2.2 *Let* $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ *be a metric nilpotent Lie algebra. A* metric solvable extension of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ *is a metric solvable Lie algebra* $(\mathfrak{s} = \mathfrak{n}\oplus\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ *such that* $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ *and* $\langle\cdot,\cdot\rangle_{\mathfrak{s}}|_{\mathfrak{n}\times\mathfrak{n}} = \langle\cdot,\cdot\rangle$. *The metric solvable Lie algebra* $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ *is* standard, *or has* standard type, *if* $\mathfrak{a}$ *is an abelian subalgebra of* $\mathfrak{s}$; *in this case, the dimension of* $\mathfrak{a}$ *is called the* rank of the metric solvable extension.

Heber showed in [15] that a simply connected solvable Lie group admits at most one Einstein left invariant metric up to isometry and scaling. Moreover, he proved that the study of Einstein metrics on simply connected solvable Lie groups, of standard type, can be reduced to the rank-one case, that is, $dim\, \mathfrak{a} = 1$.

Recently, Lauret in [20,21] proved the following

Theorem 2.3 ([20,21]) *Any Einstein metric solvable Lie algebra* $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ *has to be of standard type. Moreover, a simply connected nilpotent Lie group* N *admits a nilsoliton metric if and only if its Lie algebra* $\mathfrak{n}$ *is an Einstein nilradical, that is,* $\mathfrak{n}$ *possesses an inner product* $\langle\cdot,\cdot\rangle$ *such that* $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ *has a metric solvable extension which is Einstein.*

3 Nilsoliton Metrics Determined by Closed G_2 Forms

In this section we prove that, up to isomorphism, there are only four (nonabelian) s-step nilpotent Lie groups ($s = 2, 3$) with a nilsoliton inner product determined by a left invariant closed G_2-structure. We also show that, up to isomorphism, there is a unique 7-dimensional nilpotent Lie group with a left invariant closed G_2-structure but not having nilsoliton metrics.

Let N be a 7-dimensional simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Then, a G_2-structure on N is left invariant if and only if the corresponding 3-form is left invariant. Thus, a left invariant G_2-structure on N corresponds to an element φ of $\Lambda^3(\mathfrak{n}^*)$ that can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245},$$

with respect to some coframe $\{e^1, \ldots, e^7\}$ on $\mathfrak{n}^*$, and we shall say that φ defines a G_2-structure on $\mathfrak{n}$. A G_2-structure on $\mathfrak{n}$ is said to be *closed* if φ is closed, i.e.,

$$d\varphi = 0,$$

where d denotes the Chevalley–Eilenberg differential on $\mathfrak{n}^*$.

From now on, given a 7-dimensional Lie algebra $\mathfrak{n}$ whose dual is spanned by $\{e^1, \ldots, e^7\}$, we will write $e^{ij} = e^i \wedge e^j$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, and so forth. Moreover, by the notation

$$\mathfrak{n} = (0, 0, 0, 0, e^{12}, e^{13}, 0),$$

we mean that the dual space $\mathfrak{n}^*$ of the Lie algebra $\mathfrak{n}$ has a fixed basis $\{e^1, \ldots, e^7\}$ such that

$$de^5 = e^{12}, \quad de^6 = e^{13}, \quad de^1 = de^2 = de^3 = de^4 = de^7 = 0.$$

The classification of nilpotent Lie algebras admitting a closed G_2-structure is given in [8] as follows.

Theorem 3.1 *Up to isomorphism, there are exactly* 12 *nilpotent Lie algebras that admit a closed* G_2*-structure. They are:*

$$\begin{aligned}
\mathfrak{n}_1 &= (0, 0, 0, 0, 0, 0, 0),\\
\mathfrak{n}_2 &= (0, 0, 0, 0, e^{12}, e^{13}, 0),\\
\mathfrak{n}_3 &= (0, 0, 0, e^{12}, e^{13}, e^{23}, 0),\\
\mathfrak{n}_4 &= (0, 0, e^{12}, 0, 0, e^{13} + e^{24}, e^{15}),\\
\mathfrak{n}_5 &= (0, 0, e^{12}, 0, 0, e^{13}, e^{14} + e^{25}),\\
\mathfrak{n}_6 &= (0, 0, 0, e^{12}, e^{13}, e^{14}, e^{15}),\\
\mathfrak{n}_7 &= (0, 0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{15}),\\
\mathfrak{n}_8 &= (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34}),\\
\mathfrak{n}_9 &= (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25}),
\end{aligned}$$

$$\mathfrak{n}_{10} = (0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23}),$$
$$\mathfrak{n}_{11} = (0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26}),$$
$$\mathfrak{n}_{12} = (0, 0, 0, e^{12}, e^{23}, -e^{13}, 2e^{26} - 2e^{34} - 2e^{16} + 2e^{25}).$$

Using Table 1 in [9] we can determine which indecomposable Lie algebras $\mathfrak{n}_i$ $(4 \leq i \leq 12)$ do not have nilsoliton inner products. Note that the existence of nilsolitons on $\mathfrak{n}_2$ and $\mathfrak{n}_3$ is not studied in [9] since they are decomposable. Concretely, the correspondence between the indecomposable Lie algebras of Theorem 3.1 and Table 1 in [9] is the following:

$$\mathfrak{n}_4 \cong 3.8, \quad \mathfrak{n}_5 \cong 3.11 \quad \mathfrak{n}_6 \cong 3.20, \quad \mathfrak{n}_7 \cong 2.39,$$
$$\mathfrak{n}_8 \cong 2.5, \quad \mathfrak{n}_9 \cong 1.1(iv), \quad \text{and} \quad \mathfrak{n}_{10} \cong 1.3(i_1).$$

Moreover, $\mathfrak{n}_{11}$ and $\mathfrak{n}_{12}$ are respectively isomorphic to the real form of $1.2(i_{-3})$ and $3.1(i_2)$. In particular, we have that $\mathfrak{n}_9$ is the only 7-dimensional nilpotent Lie algebra with a closed G_2 form but not admitting a nilsoliton.

Remark 3.2 Note that the abelian Lie algebra $\mathfrak{n}_1$ admits as rank-one Einstein solvable extension the Lie algebra $\mathfrak{s}_1$ with structure equations

$$(ae^{18}, ae^{28}, ae^{38}, ae^{48}, ae^{58}, ae^{68}, ae^{78}, 0),$$

for some real number $a \neq 0$, and the nilsoliton inner product on $\mathfrak{n}_1$ is trivial because it is flat. Since we are interested in nontrivial nilsolitons inner products, in the sequel when we refer to a nilpotent Lie algebra we will mean a nonabelian nilpotent Lie algebra.

In order to classify the Lie algebras $\mathfrak{n}_i$ admitting a (nontrivial) nilsoliton but with no closed G_2 forms inducing the nilsoliton, we need to recall the following obstruction proved in [8] for the existence of a closed G_2-structure on a 7-dimensional Lie algebra.

Lemma 3.3 ([8]). *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra. If there is a non-zero $X \in \mathfrak{g}$ such that $(\iota_X\phi)^3 = 0$ (where ι_X denotes the contraction by X) for every closed 3-form ϕ on $\mathfrak{g}$, then $\mathfrak{g}$ has no closed G_2-structures.*

By [28, Proposition 4.5] if φ is a G_2-structure on a 7-dimensional Lie algebra and we choose a vector $X \in \mathfrak{g}$ of length one with respect to g_φ, then on the orthogonal complement of the span of X one has an $SU(3)$-structure given by the 2-form $\alpha = \iota_X\varphi$ and the 3-form $\beta = \varphi - \alpha \wedge \eta$, where $\eta = \iota_X(g_\varphi)$. So in particular $\alpha \wedge \beta = 0$.

By using these results we can prove the following proposition

Proposition 3.4 *The Lie algebra $\mathfrak{n}_i$ $(i = 3, 5, 7, 8, 11)$ has a nilsoliton inner product but no closed G_2-structure inducing the nilsoliton inner product.*

Proof To prove that $\mathfrak{n}_3$ has a nilsoliton, we consider the Lie algebra $\mathfrak{n}_3$ defined by the equations given in Theorem 3.1. Let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_3}$ be the inner product on $\mathfrak{n}_3$ such that $\{e^1, \ldots, e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_3}$ is a nilsoliton because its Ricci tensor

$$Ric = diag\left(-1, -1, -1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)$$

satisfies (3), for $\lambda = -5/2$ and

$$D = diag\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 3, 3, 3, \frac{5}{2}\right).$$

Since the nilsoliton inner product is unique (up to isometry and scaling) it suffices to prove that there is no closed G_2 form on $\mathfrak{n}_3$ inducing such an inner product.

Suppose that $\mathfrak{n}_3$ has a closed G_2 form ϕ such that

$$g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_3} = \sum_{i=1}^{7}(e^i)^2. \tag{4}$$

Thus, g_ϕ has to satisfy

$$\prod_{i=1}^{7} g_\phi(e_i, e_i) = 1. \tag{5}$$

A generic closed 3-form γ on $\mathfrak{n}_3$ has the following expression

$$\begin{aligned}\gamma = {} & c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135}\\ & + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} + c_{157}e^{157}\\ & + c_{167}e^{167} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{237}e^{237} + c_{146}e^{245} + c_{246}e^{246}\\ & + c_{247}e^{247} + c_{256}e^{256} + c_{257}e^{257} + c_{267}e^{267} + c_{156}e^{345} + c_{256}e^{346}\\ & + (c_{257} - c_{167})\, e^{347} + c_{356}e^{356} + c_{357}e^{357} + c_{367}e^{367},\end{aligned}$$

where c_{ijk} are arbitrary real numbers.

Now, we show conditions on the coefficients c_{ijk} so that $\phi = \gamma$ is a closed G_2 form such that g_ϕ satisfies (4). To this end, we apply the aforementioned result of [28, Proposition 4.5] for $X = e_i$ $(1 \le i \le 7)$ and so $\eta = e^i$ by (4). For $X = e_1$, thus $\eta = e^1$, we have

$$\begin{aligned}\alpha_1 = \iota_{e_1}\phi = {} & c_{123}e^{23} + c_{124}e^{24} + c_{125}e^{25} + c_{126}e^{26} + c_{127}e^{27} + c_{134}e^{34} + c_{135}e^{35}\\ & + c_{136}e^{36} + c_{137}e^{37} + c_{145}e^{45} + c_{146}e^{46} + c_{147}e^{47} + c_{156}e^{56} + c_{157}e^{57} + c_{167}e^{67},\end{aligned}$$

and

$$\begin{aligned}\beta_1 = \phi - \iota_{e_1}\phi \wedge e^1 = {} & c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{237}e^{237} + c_{146}e^{245} + c_{246}e^{246}\\ & + c_{247}e^{247} + c_{256}e^{256} + c_{257}e^{257} + c_{267}e^{267} + c_{156}e^{345} + c_{256}e^{346}\\ & + (c_{257} - c_{167})\, e^{347} + c_{356}e^{356} + c_{357}e^{357} + c_{367}e^{367},\end{aligned}$$

But, $\alpha_1 \wedge \beta_1 = 0$ describes a system of 6 equations. Hence, after apply the result of [28, Proposition 4.5] for $X = e_2, \ldots, e_7$, we obtain a system of 42 equations. This system and condition (4) imply that any closed G_2 form on $\mathfrak{n}_3$ satisfying (5) is expressed as follows

$$\begin{aligned}\phi = {} & c_{123}e^{123} + c_{145}e^{145} + c_{167}e^{167} + c_{246}e^{246} + c_{257}e^{257}\\ & + (c_{257} - c_{167})\, e^{347} + c_{356}e^{356}.\end{aligned} \tag{6}$$

Because ϕ should be a closed G_2 form on $\mathfrak{n}_3$, at least for certain coefficients c_{ijk}, Lemma 3.3 implies that the coefficients appearing on (6) cannot vanish. In particular, $c_{257} - c_{167} \neq 0$. Now, denote by G_ϕ the matrix associated with the inner product on $\mathfrak{n}_3$ induced by the 3-form ϕ given by (6). Then, (4) implies that $G_\phi = I_7$, for some c_{ijk} and then

$$S = G_\phi - I_7 = 0, \qquad (7)$$

for those coefficients. From now on, we denote by S_{ij} the (i, j) entry of the matrix S. One can check that the equations $S_{11} = S_{22} = S_{55} = 0$ imply that

$$c_{356} = \frac{1}{c_{145}c_{257}}, \quad c_{246} = -\frac{c_{145}c_{167}}{c_{257}} \quad \text{and} \quad c_{123} = \frac{1}{c_{145}c_{167}}.$$

Therefore, the expression of S_{66} becomes

$$S_{66} = \frac{(c_{167} - c_{257})(c_{167} + c_{257})}{c_{257}^2},$$

and hence $c_{167} = \pm c_{257}$. But we know that $c_{167} \neq c_{257}$, and for $c_{167} = -c_{257}$, we have that $S_{33} = -c_{123}(c_{167} - c_{257})c_{356}$ and so $S \neq 0$, which is a contradiction with (7). This means that $\mathfrak{n}_3$ does not admit a closed G_2 form inducing the nilsoliton given by (4).

To prove that $\mathfrak{n}_5$ has a nilsoliton, we consider the Lie algebra $\mathfrak{n}_5$ defined by the structure equations

$$\mathfrak{n}_5 = \left(0, 0, \sqrt{3}e^{12}, 0, 0, 2e^{13}, e^{14} + \sqrt{3}e^{25}\right).$$

Consider the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}_5}$ such that the basis $\{e^1, \ldots, e^7\}$ is orthonormal. Then, its Ricci tensor satisfies

$$Ric = diag\left(-4, -3, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, 2, 2\right).$$

Actually, $Ric = -\frac{13}{2}I_7 + D$, where D is the derivation of $\mathfrak{n}_5$ given by

$$D = diag\left(\frac{5}{2}, \frac{7}{2}, 6, 6, 5, \frac{17}{2}, \frac{17}{2}\right)$$

and so $\langle \cdot, \cdot \rangle_{\mathfrak{n}_5} = \sum_{i=1}^{7}(e^i)^2$ is a nilsoliton inner product.

Since the nilsoliton inner product is unique (up to isometry and scaling) it is sufficient to prove that there is no closed G_2 form on $\mathfrak{n}_5$ inducing such an inner product. Suppose that $\mathfrak{n}_5$ has a closed G_2 form ϕ such that $g_\phi = \langle \cdot, \cdot \rangle_{\mathfrak{n}_5}$.

A generic closed 3-form γ on $\mathfrak{n}_5$ has the following expression

$$\gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} + c_{157}e^{157} + c_{167}e^{167} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{237}e^{237} + c_{245}e^{245} + \frac{1}{2}c_{237}e^{246} + c_{247}e^{247} - \frac{1}{2}\sqrt{3}c_{137}e^{256} + \sqrt{3}(c_{345} - c_{147})e^{257} + c_{345}e^{345} - c_{167}e^{356} + c_{457}e^{457},$$

where c_{ijk} are arbitrary real numbers. Now we show conditions on the coefficients c_{ijk} so that $\phi = \gamma$ is a closed G_2 form such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_5}$. Lemma 3.3 (applied for $X = e_7$) implies that

$$c_{167}\, c_{237}\, c_{457} \neq 0. \tag{8}$$

Now, we denote by G_ϕ the matrix associated with the inner product on $\mathfrak{n}_5$ induced by the generic closed 3-form ϕ. Then the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_5}$ implies (7) for some coefficients c_{ijk}. From the equations $S_{66} = S_{77} = S_{67} = S_{37} = S_{46} = S_{33} = S_{36} = S_{47} = 0$ we have that

$$c_{237} = \frac{2}{c_{167}^2}, \quad c_{457} = \frac{1}{2}c_{167}, \quad c_{236} = -2c_{247}, \quad c_{136} = -2c_{147},$$
$$c_{345} = 0, \qquad c_{134} = \frac{1}{2}c_{167}, \qquad c_{137} = 2c_{146}, \qquad c_{234} = 0.$$

Therefore, $S_{44} = -\frac{3}{8}c_{167}^2c_{237}$ which by (8) cannot vanish and so $S \neq 0$, which is a contradiction with (7).

Consider now the Lie algebra $\mathfrak{n}_7$ defined by the structure equations

$$\mathfrak{n}_7 = \left(0, 0, 0, e^{12}, \frac{\sqrt{6}}{2}e^{13}, e^{14} + \frac{\sqrt{6}}{2}e^{23}, \sqrt{2}e^{15}\right).$$

Let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$ be the inner product on $\mathfrak{n}_7$ such that the basis $\{e^1, \ldots, e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_7} = \sum_{i=1}^7 (e^i)^2$ is a nilsoliton since

$$Ric = \left(-\frac{11}{4}, -\frac{5}{4}, -\frac{3}{2}, 0, -\frac{1}{4}, \frac{5}{4}, 1\right) = -4I_7 + D,$$

where

$$D = diag\left(\frac{5}{4}, \frac{11}{4}, \frac{5}{2}, 4, \frac{15}{4}, \frac{21}{4}, 5\right),$$

is a derivation of $\mathfrak{n}_7$. As before, since the nilsoliton inner product is unique (up to isometry and scaling) it suffices to prove that there is no closed G_2 form on $\mathfrak{n}_7$ inducing such an inner product.

Suppose that $\mathfrak{n}_7$ has a closed G_2 form ϕ such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$. A generic closed 3-form γ on $\mathfrak{n}_7$ has the following expression

$$\begin{aligned}\gamma &= c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135}\\ &\quad + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} + c_{157}e^{157}\\ &\quad + c_{167}e^{167} + c_{234}e^{234} + c_{235}e^{235} + \left(\frac{\sqrt{6}}{2}c_{245} - \frac{\sqrt{6}}{2}c_{146}\right)e^{236} + c_{237}e^{237}\\ &\quad + c_{245}e^{245} + c_{246}e^{246} + \frac{\sqrt{2}}{2}c_{256}e^{247} + c_{256}e^{256} + \left(c_{167} + \frac{\sqrt{6}}{3}c_{347}\right)e^{257}\\ &\quad + \left(\frac{\sqrt{6}}{2}c_{156} + \sqrt{2}c_{237}\right)e^{345} + \frac{\sqrt{6}}{2}c_{256}e^{346} + c_{347}e^{347} + \sqrt{2}c_{347}e^{356} + c_{357}e^{357},\end{aligned}$$

where c_{ijk} are arbitrary real numbers. Now, we show conditions on the coefficients c_{ijk} so that $\phi = \gamma$ be a closed G_2 form such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$. Lemma 3.3 applied for $X = e_7$ implies that

$$c_{167} \neq 0. \tag{9}$$

Now we apply the result of [28, Proposition 4.5] for $X = e_i$ $(1 \le i \le 7)$ and so $\eta = e^i$ by (4). For $X = e_1$, we have

$$\begin{aligned}\alpha_1 = \iota_{e_1}\phi &= c_{123}e^{23} + c_{124}e^{24} + c_{125}e^{25} + c_{126}e^{26} + c_{127}e^{27} + c_{134}e^{34} + c_{135}e^{35}\\ &\quad + c_{136}e^{36} + c_{137}e^{37} + c_{145}e^{45} + c_{146}e^{46} + c_{147}e^{47} + c_{156}e^{56} + c_{157}e^{57}\\ &\quad + c_{167}e^{67}\end{aligned}$$

and

$$\begin{aligned}\beta_1 = \phi - \iota_{e_1}\phi \wedge e^1 &= c_{234}e^{234} + c_{235}e^{235} + \left(\frac{\sqrt{6}}{2}c_{245} - \frac{\sqrt{6}}{2}c_{146}\right)e^{236} + c_{237}e^{237}\\ &\quad + c_{245}e^{245} + c_{246}e^{246} + \frac{\sqrt{2}}{2}c_{256}e^{247} + c_{256}e^{256} + \left(c_{167} + \frac{\sqrt{6}}{3}c_{347}\right)e^{257}\\ &\quad + \left(\frac{\sqrt{6}}{2}c_{156} + \sqrt{2}c_{237}\right)e^{345} + \frac{\sqrt{6}}{2}c_{256}e^{346} + c_{347}e^{347} + \sqrt{2}c_{347}e^{356} + c_{357}e^{357}.\end{aligned}$$

Therefore, $\alpha_1 \wedge \beta_1 = 0$ describes a system of 6 equations. Hence, after apply the result of [28, Proposition 4.5] for $X = e_2, \ldots, e_7$, we obtain a system of 42 equations. This system together with the fact that $c_{167} \neq 0$ and the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$ imply that any closed G_2 form on $\mathfrak{n}_7$ satisfying (5) is expressed as follows

$$\phi = c_{123}e^{123} + c_{145}e^{145} + c_{167}e^{167} + c_{246}e^{246} + \left(c_{167} + \frac{\sqrt{6}}{3}c_{347}\right)e^{257} \\ + c_{347}e^{347} + \sqrt{2}c_{347}e^{356}. \tag{10}$$

Now we denote by G_ϕ the matrix associated with the inner product on $\mathfrak{n}_7$ induced by the 3-form ϕ given by (10). Then, the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$ implies (7) is satisfied for some coefficients c_{ijk}. From equations $S_{11} = S_{33} = S_{44} = S_{66} = 0$ we have

$$c_{123} = \frac{\sqrt{2}}{2c_{347}^3}, \quad c_{145} = -\sqrt{2}c_{347}, \quad c_{167} = -c_{347}, \quad \text{and} \quad c_{246} = \frac{\sqrt{2}}{2c_{347}^3}.$$

Therefore $S_{55} = 1$ and so $S \neq 0$ which is a contradiction with (7).

Let $\mathfrak{n}_8$ be the Lie algebra described by the structure equations

$$\mathfrak{n}_8 = \left(0, 0, e^{12}, -e^{13}, -e^{23}, e^{15} + e^{24}, -e^{16} - e^{34}\right),$$

and let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_8}$ be the inner product on $\mathfrak{n}_8$ such that $\{e^1, \dots, e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_8} = \sum_{i=1}^7 (e^i)^2$ is a nilsoliton because its Ricci tensor

$$Ric = diag\left(-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right)$$

satisfies (3), for $\lambda = -\frac{5}{2}$ and

$$D = diag\left(\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\right).$$

The nilsoliton inner product is unique (up to isometry and scaling) therefore it suffices to prove that there is no closed G_2 form on $\mathfrak{n}_8$ inducing such an inner product. A generic closed 3-form γ on $\mathfrak{n}_8$ has the following expression

$$\begin{aligned}\gamma = {} & c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{1,2,7}e^{127} + c_{134}e^{134} + c_{135}e^{135} \\ & + c_{136}e^{136} + c_{137}e^{137} + (-c_{127} - c_{136})\, e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} \\ & + c_{157}e^{157} + c_{234}e^{234} + c_{235}e^{235} + c_{2,3,6}e^{236} + c_{237}e^{237} + c_{236}e^{245} \\ & + (c_{156} - c_{237})\, e^{246} + c_{157}e^{247} + c_{256}e^{256} + c_{267}e^{267} + (c_{237} - 2c_{156})\, e^{345} \\ & + c_{157}e^{346} - c_{267}e^{357} + c_{267}e^{456},\end{aligned}$$

where c_{ijk} are real numbers. Now, we show conditions on the coefficients c_{ijk} so that $\phi = \gamma$ is a closed G_2 form such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_8}$. We apply the result previously mentioned [28, Proposition 4.5] for $X = e_i$ $(1 \le i \le 7)$ and so $\eta = e^i$ by the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_8}$. After solving the system of 42 equations we have that any closed G_2 form on $\mathfrak{n}_8$ satisfying (5) is expressed as follows

$$\phi = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{135}e^{135} + c_{136}e^{136} - c_{136}e^{145} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{236}e^{245} + c_{256}e^{256}. \tag{11}$$

Now denote by G_ϕ the matrix associated with the inner product on $\mathfrak{n}_8$ induced by the 3-form ϕ given by (11). Then $G_\phi = 0$ obtaining a contradiction with (7).

It only remains to study the Lie algebra $\mathfrak{n}_{11}$. According to Theorem 3.1, $\mathfrak{n}_{11}$ is defined by the equations

$$\mathfrak{n}_{11} = \left(0, 0, f^{12}, 0, f^{13}, f^{24} + f^{23}, f^{25} + f^{34} + f^{15} + f^{16} - 3f^{26}\right).$$

We consider the new basis $\{e^j\}_{j=1}^7$ of $\mathfrak{n}_{11}^*$ with

$$\left\{e^1 = f^2, e^2 = -\frac{\sqrt{3}}{3}f^1, e^3 = \frac{\sqrt{39}}{39}f^3 + \frac{\sqrt{39}}{78}f^4, e^4 = -\frac{\sqrt{78}}{78}f^4, e^5 = \frac{\sqrt{3}}{39}f^6, e^6 = -\frac{1}{3}f^5, e^7 = -\frac{\sqrt{3}}{1014}f^7\right\}.$$

Thus, the Lie algebra $\mathfrak{n}_{11}$ can also be described by the structure equations

$$\mathfrak{n}_{11} = \Bigg(0, 0, \frac{\sqrt{13}}{13}e^{12}, 0, \frac{\sqrt{13}}{13}e^{13} - \frac{\sqrt{26}}{26}e^{14}, \frac{\sqrt{26}}{26}e^{24} + \frac{\sqrt{13}}{13}e^{23}, \frac{\sqrt{13}}{26}e^{25} + \frac{\sqrt{26}}{26}e^{34} + \frac{\sqrt{39}}{26}e^{15} + \frac{\sqrt{13}}{26}e^{16} - \frac{\sqrt{39}}{26}e^{26}\Bigg),$$

Let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$ be the inner product on $\mathfrak{n}_{11}$ such that $\{e^1,\ldots,e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}} = \sum_{i=1}^7 (e^i)^2$ is a nilsoliton because its Ricci tensor

$$Ric = \frac{1}{52}diag(-7, -7, -3, -3, 1, 1, 5)$$

satisfies $Ric = -\frac{11}{52}Id + D$, where D is the derivation of the Lie algebra $\mathfrak{n}_{11}$ given by

$$D = \frac{1}{13}diag(1, 1, 2, 2, 3, 3, 4).$$

It suffices to prove that there is no closed G_2 form on $\mathfrak{n}_{11}$ inducing such an inner product. Let's suppose that $\mathfrak{n}_{11}$ has a closed G_2 form ϕ such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$. A generic closed 3-form γ on $\mathfrak{n}_{11}$ has the following expression

$$\gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} - \sqrt{\tfrac{3}{2}}c_{347}e^{157} + \frac{c_{347}e^{167}}{\sqrt{2}} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + \left(\frac{c_{137}}{\sqrt{3}} - \frac{2c_{156}}{\sqrt{3}}\right)e^{237}$$

$$+\left(\frac{c_{127}}{\sqrt{2}}-\frac{c_{136}}{\sqrt{2}}+c_{146}-\frac{c_{235}}{\sqrt{2}}\right)e^{245}+c_{246}e^{246}+c_{247}e^{247}+\left(\frac{c_{156}}{\sqrt{3}}-\frac{2c_{137}}{\sqrt{3}}\right)e^{256}$$
$$+\frac{c_{347}e^{257}}{\sqrt{2}}+\sqrt{\tfrac{3}{2}}c_{347}e^{267}+\left(\tfrac{1}{2}c_{147}-\frac{c_{156}}{\sqrt{2}}-\tfrac{1}{2}\sqrt{3}c_{247}\right)e^{345}$$
$$+\left(-\sqrt{\tfrac{2}{3}}c_{137}-\tfrac{1}{2}\sqrt{3}c_{147}+\frac{c_{156}}{\sqrt{6}}-\tfrac{1}{2}c_{247}\right)e^{346}+c_{347}e^{347}+\sqrt{2}c_{347}e^{356}$$

where c_{ijk} are arbitrary real numbers.

Now, we show conditions on the coefficients c_{ijk} so that $\phi=\gamma$ is a closed G_2 form such that $g_\phi=\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$. We apply the result of [28, Proposition 4.5] for $X=e_i$ $(1\le i\le 7)$ and so $\eta=e^i$ by the condition $g_\phi=\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$. After solving the system of 42 equations we have that any closed G_2 form on $\mathfrak{n}_{11}$ satisfying (5) is expressed as follows

$$\phi=c_{123}e^{123}-c_{246}e^{145}-\sqrt{3}c_{246}e^{167}-\frac{\sqrt{6}}{2}c_{347}e^{157}+\frac{\sqrt{2}}{2}c_{347}e^{167}-\sqrt{3}c_{246}e^{245}$$
$$+c_{246}e^{246}+\frac{\sqrt{2}}{2}c_{347}e^{257}+\frac{\sqrt{6}}{2}c_{347}e^{267}+c_{347}e^{347}+\sqrt{2}c_{347}e^{356}. \tag{12}$$

As before denote by G_ϕ the matrix associated with the inner product induced by the 3-form ϕ given by (12). Then, the condition $g_\phi=\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$ implies (7), for some c_{ijk}. Equations $S_{66}=S_{77}=0$ imply that

$$c_{246}=-\frac{1}{2}c_{347},\quad\text{and}\quad c_{347}=2^{-1/3}.$$

Therefore, $S_{44}=-\frac{1}{2}$ and so $S\neq 0$ which contradicts (7). □

Remark 3.5 Note that the 4-step nilpotent Lie algebra $\mathfrak{n}_{10}$ is isomorphic in the classification given in [11] to the Lie algebra $1.3(i)[\lambda=1]$ and the existence of the nilsoliton was shown in [9, Example 2]. Since an explicit expression of the nilsoliton is not known, we cannot apply the argument used in the proof of Proposition 3.4. Thus, it remains open the question of whether the Lie algebra $\mathfrak{n}_{10}$ admits a closed G_2 form inducing a nilsoliton or not. Moreover, the explicit expression of the nilsolitons for $\mathfrak{n}_{11}$ and $\mathfrak{n}_{12}$ have been already determined in [11] (see there page 20, Remark 3.5), but our basis is different for the nilsoliton on the other Lie algebras.

Theorem 3.6 *Up to isomorphism,* $\mathfrak{n}_2$*,* $\mathfrak{n}_4$*,* $\mathfrak{n}_6$ *and* $\mathfrak{n}_{12}$ *are the unique* s*-step nilpotent Lie algebras* $(s=2,3)$ *with a nilsoliton inner product determined by a closed* G_2*-structure.*

Proof We will show that the Lie algebra $\mathfrak{n}_i$ $(i=2,4,6,12)$ has a closed G_2 form φ_i such that the Ricci tensor of the inner product g_{φ_i} satisfies (3), for some derivation D of $\mathfrak{n}_i$ and some real number λ.

For $\mathfrak{n}_2$ we consider the closed G_2 form φ_2 defined by

$$\varphi_2=e^{147}+e^{267}+e^{357}+e^{123}+e^{156}+e^{245}-e^{346}. \tag{13}$$

The inner product g_{φ_2} given by (1) is the one making orthonormal the basis $\{e^1, \dots, e^7\}$, and it is a nilsoliton since $Ric = -2I_7 + D$, where

$$D = diag\left(1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2\right)$$

is a derivation of $\mathfrak{n}_2$.

On the Lie algebra $\mathfrak{n}_4$, we define the G_2 form φ_4 by

$$\varphi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236}. \tag{14}$$

Then, φ_4 is closed, the inner product g_{φ_4} makes the basis $\{e^1, \dots, e^7\}$ orthonormal and g_{φ_4} is a nilsoliton since $Ric = -\frac{5}{2}I_7 + D$, where D is the derivation of $\mathfrak{n}_4$ given by

$$D = diag\left(1, \frac{3}{2}, \frac{5}{2}, 2, 2, \frac{7}{2}, 3\right).$$

For the Lie algebra $\mathfrak{n}_6$ we consider the closed G_2-structure defined by the 3-form

$$\varphi_6 = e^{123} + e^{145} + e^{167} + e^{257} - e^{246} + e^{347} + e^{356}. \tag{15}$$

Therefore, the inner product g_{φ_6} is such that the basis $\{e^1, \dots, e^7\}$ is orthonormal and it is a nilsoliton since $Ric = -\frac{5}{2}I_7 + D$, where D is the derivation of $\mathfrak{n}_6$ given by

$$D = diag\left(\frac{1}{2}, 2, 2, \frac{5}{2}, \frac{5}{2}, 3, 3\right).$$

Theorem 3.1 implies that the Lie algebra $\mathfrak{n}_{12}$ is defined by the equations

$$\mathfrak{n}_{12} = \left(0, 0, 0, h^{12}, h^{23}, -h^{13}, 2h^{26} - 2h^{34} - 2h^{16} + 2h^{25}\right).$$

We consider the basis $\{e^i\}_{i=1}^7$ of $\mathfrak{n}_{12}^*$ given by

$$\Big\{e^1 = \frac{\sqrt{3}}{2}h^2, e^2 = h^1 - \frac{1}{2}h^2, e^3 = h^3, e^4 = -\frac{1}{4}h^4, e^5 = \frac{1}{4}h^5 + \frac{1}{4}h^6,$$
$$e^6 = -\frac{\sqrt{3}}{12}h^5 + \frac{\sqrt{3}}{12}h^6, e^7 = -\frac{\sqrt{3}}{48}h^7\Big\}.$$

Then, $\mathfrak{n}_{12}$ is defined as follows

$$\begin{aligned}\mathfrak{n}_{12} = \Big(0, 0, 0, \frac{\sqrt{3}}{6}e^{12}, -\frac{1}{4}e^{23} + \frac{\sqrt{3}}{12}e^{13}, -\frac{\sqrt{3}}{12}e^{23} - \frac{1}{4}e^{13},\\ -\frac{\sqrt{3}}{6}e^{34} + \frac{\sqrt{3}}{12}e^{25} + \frac{1}{4}e^{26} + \frac{\sqrt{3}}{12}e^{16} - \frac{1}{4}e^{15}\Big).\end{aligned} \tag{16}$$

We define the G_2 form φ_{12} by

$$\varphi_{12} = -e^{124} + e^{135} + e^{167} - e^{236} + e^{257} + e^{347} - e^{456}. \tag{17}$$

Clearly φ_{12} is closed. Moreover, φ_{12} defines the inner product $g_{\varphi_{12}}$ which makes the basis $\{e^1, \ldots, e^7\}$ orthonormal, and $g_{\varphi_{12}}$ is a nilsoliton since $Ric = -\frac{1}{4}Id + \frac{1}{8}D$, where D is the derivation of $\mathfrak{n}_{12}$ given by

$$D = diag(1, 1, 1, 2, 2, 2, 3).$$

□

4 Laplacian Flow

Let us consider the nilpotent Lie algebra $\mathfrak{n}_i$ $(i = 2, 4, 6)$ defined in Theorem 3.1, and the Lie algebra $\mathfrak{n}_{12}$ defined by (16). Let N_i be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}_i$, and let φ_i be the closed G_2 form on N_i $(i = 2, 4, 6, 12)$ given by (13), (14), (15) and (17), for $i = 2, 4, 6$ and 12, respectively.

The purpose of this section is to prove long time existence and uniqueness of solution for the Laplacian flow of φ_i on N_i, and that the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in N_i, as t goes to infinity.

Let M be a 7-dimensional manifold with an arbitrary G_2 form φ. The *Laplacian flow* of φ is defined to be

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = \Delta_t \varphi(t), \\ \varphi(0) = \varphi, \end{cases}$$

where Δ_t is the Hodge Laplacian of the metric g_t determined by the G_2 form $\varphi(t)$.

For the different types of G_2-structures the behavior of the solution of the Laplacian flow is very different. For example, the stable solutions of the Laplacian flow are given by the G_2 manifolds (M, φ) such that $Hol(M) \subseteq G_2$.

The study of the Laplacian flow of a closed G_2 form φ on a manifold M consists to study long time existence, convergence and formation of singularities for the system of differential equations

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = \Delta_t \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi. \end{cases} \tag{18}$$

In the case of closed G_2-structures on compact manifolds, Bryant and Xu [4] gave a result of short time existence and uniqueness of solution.

Theorem 4.1 *[4] If M is compact, then* (18) *has a unique solution for a short time $0 \leq t < \epsilon$, with ϵ depending on $\varphi = \varphi(0)$.*

In the following theorem we determine a global solution of the Laplacian flow of the closed G_2 form φ_2 on N_2.

Theorem 4.2 *The family of closed G_2 forms $\varphi_2(t)$ on N_2 given by*

$$\varphi_2(t) = e^{147} + e^{267} + e^{357} + f(t)^3 e^{123} + e^{156} + e^{245} - e^{346}, \qquad t \in \left(-\frac{3}{10}, +\infty\right), \tag{19}$$

is the solution of the Laplacian flow (18) of φ_2, where $f = f(t)$ is the function

$$f(t) = \left(\frac{10}{3}t + 1\right)^{\frac{1}{5}}.$$

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in N_2, as t goes to infinity.

Proof Let $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is a real open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_2 defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 7.$$

From now on we write $f_{ij} = f_{ij}(t) = f_i(t)f_j(t)$, $f_{ijk} = f_{ijk}(t) = f_i(t)f_j(t)f_k(t)$, and so forth. Then, the structure equations of N_2 with respect to this basis are

$$dx^i = 0, \quad i = 1, 2, 3, 4, 7, \qquad dx^5 = \frac{f_5}{f_{12}}x^{12}, \qquad dx^6 = \frac{f_6}{f_{13}}x^{13}. \tag{20}$$

Now, for any $t \in I$, we consider the G_2 form $\varphi_2(t)$ on N_2 given by

$$\begin{aligned}\varphi_2(t) &= x^{147} + x^{267} + x^{357} + x^{123} + x^{156} + x^{245} - x^{346} \\ &= f_{147}e^{147} + f_{267}e^{267} + f_{357}e^{357} + f_{123}e^{123} + f_{156}e^{156} + f_{245}e^{245} - f_{346}e^{346}.\end{aligned} \tag{21}$$

Note that $\varphi_2(0) = \varphi_2$ and, for any t, the 3-form $\varphi_2(t)$ on N_2 determines the metric g_t such that the basis $\{x_i = \frac{1}{f_i}e_i;\ i = 1, \ldots, 7\}$ of $\mathfrak{n}_2$ is orthonormal. So, $g(t)(e_i, e_i) = f_i{}^2$.

Using (20), one can check that $d\varphi_2(t) = 0$ if and only if

$$f_{26}(t) = f_{35}(t), \tag{22}$$

for any t. Assuming $f_i(0) = 1$ and (22), to solve the flow (18) of φ_2, we need to determine the functions f_i and the interval I so that $\frac{d}{dt}\varphi_2(t) = \Delta_t \varphi_2(t)$, for $t \in I$. Using (21) we have

$$\frac{d}{dt}\varphi_2(t) = \left(f_{147}\right)' e^{147} + \left(f_{267}\right)' e^{267} + \left(f_{357}\right)' e^{357} + \left(f_{123}\right)' e^{123} + \left(f_{156}\right)' e^{156} + \left(f_{245}\right)' e^{245} - \left(f_{346}\right)' e^{346}. \tag{23}$$

Now, we calculate $\Delta_t \varphi_2(t) = -d *_t d *_t \varphi_2(t)$. On the one hand, we have

$$*_t\varphi_2(t) = x^{2356} - x^{1345} - x^{1246} + x^{4567} + x^{2347} - x^{1367} + x^{1257}. \tag{24}$$

So, x^{4567} is the unique nonclosed summand in $*_t\varphi_2(t)$. Then, taking into account (22), we obtain

$$d(*_t d *_t \varphi_2(t)) = \frac{f_6}{f_{13}}\left(-\frac{f_6}{f_{13}}x^{123} - \frac{f_5}{f_{12}}x^{123}\right) = -2\left(\frac{f_6}{f_{13}}\right)^2 x^{123}.$$

Therefore, in terms of the forms e^{ijk}, the expression of $-d(*_t d *_t \varphi_2(t))$ is

$$-d(*_t d *_t \varphi_2(t)) = 2 f_{123}\left(\frac{f_6}{f_{13}}\right)^2 e^{123} = 2\left(\frac{f_2(f_6)^2}{f_{13}}\right) e^{123}. \tag{25}$$

Comparing (23) and (25) we see that, in particular, $f_{156}(t) = 1$, for any $t \in I$. Then, using (22), we have

$$\frac{f_2(f_6)^2}{f_{13}} = \frac{1}{(f_1)^2}.$$

This equality and (25) imply that $-d(*_t d *_t \varphi_2(t))$ can be expressed as follows

$$-d(*_t d *_t \varphi_2(t)) = 2\frac{1}{(f_1)^2} e^{123}. \tag{26}$$

Then, from (23) and (26) we have that $\frac{d}{dt}\varphi_2(t) = \Delta_t\varphi_2(t)$ if and only if the functions $f_i(t)$ satisfy the following system of differential equations

$$\begin{aligned}&\left(f_{147}\right)' = \left(f_{267}\right)' = \left(f_{357}\right)' = \left(f_{156}\right)' = \left(f_{245}\right)' = \left(f_{346}\right)' = 0,\\ &\left(f_{123}\right)' = 2\frac{1}{(f_1)^2}.\end{aligned} \tag{27}$$

Because $\varphi_2(0) = \varphi_2$, the equations in the first line of (27) imply

$$f_{147}(t) = f_{267}(t) = f_{357}(t) = f_{156}(t) = f_{245}(t) = f_{346}(t) = 1, \tag{28}$$

for any $t \in I$. From the equations (28) we obtain

$$f_1^2 = f_2^2 = f_3^2.$$

Let us consider $f = f_1 = f_2 = f_3$. Using again (28) we have

$$f_i(t) = \left(f(t)\right)^{-\frac{1}{2}}, \quad i = 4, 5, 6, 7.$$

Now, the last equation of (27) implies that $f^4 f' = \frac{2}{3}$. Integrating this equation, we obtain

$$f^5 = \frac{10}{3}t + B, \quad B = constant.$$

But $\varphi_2(0) = \varphi_2$ implies $f^3(0) = f_{123}(0) = 1$, that is, $B = 1$. Hence,

$$f(t) = \left(\frac{10}{3}t + 1\right)^{\frac{1}{5}},$$

and so the one-parameter family of 3-forms $\{\varphi_2(t)\}$ given by (19) is the solution of the Laplacian flow of φ_2 on N_2, and it is defined for every $t \in (-\frac{3}{10}, +\infty)$.

To complete the proof, we study the behavior of the underlying metric $g(t)$ of such a solution in the limit for $t \to +\infty$. Indeed, if we think of the Laplacian flow as a one-parameter family of G_2 manifolds with a closed G_2-structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. For every $t \in \left(-\frac{3}{10}, +\infty\right)$, denote by $g(t)$ the metric on N_2 induced by the G_2 form $\varphi_2(t)$ given by (19). Then,

$$\begin{aligned} g(t) = {} & \left(\frac{10}{3}t + 1\right)^{2/5} (e^1)^2 + \left(\frac{10}{3}t + 1\right)^{2/5} (e^2)^2 + \left(\frac{10}{3}t + 1\right)^{2/5} (e^3)^2 \\ & + \left(\frac{10}{3}t + 1\right)^{-1/5} (e^4)^2 + \left(\frac{10}{3}t + 1\right)^{-1/5} (e^5)^2 + \left(\frac{10}{3}t + 1\right)^{-1/5} (e^6)^2 \\ & + \left(\frac{10}{3}t + 1\right)^{-1/5} (e^7)^2. \end{aligned}$$

Concretely, taking into account the symmetry properties of the Riemannian curvature $R(t)$ we obtain

$$\begin{aligned} & R_{1212} = R_{1313} = -\frac{3}{4(1 + \frac{10}{3}t)}, \\ & R_{1515} = R_{1616} = R_{3636} = R_{2525} = \frac{1}{4(1 + \frac{10}{3}t)}, \\ & R_{2356} = -\frac{1}{4(1 + \frac{10}{3}t)}, \quad R_{ijkl} = 0 \quad \text{otherwise}, \end{aligned}$$

where $R_{ijkl} = R(t)(e_i, e_j, e_k, e_l)$. Therefore, $\lim_{t \to +\infty} R(t) = 0$. □

Remark 4.3 Note that, for every $t \in \left(-\frac{3}{10}, +\infty\right)$, the metric $g(t)$ is a nilsoliton on the Lie algebra $\mathfrak{n}_2$ of N_2. In fact, with respect to the orthonormal basis $(x_1(t), \ldots, x_7(t))$, we have

$$\begin{aligned} Ric(g(t)) &= -\frac{6}{(3+10t)} Id + \frac{3}{(3+10t)} diag\left(1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2\right) \\ &= \frac{3}{(3+10t)} Ric(g(0)) \end{aligned}$$

with $\frac{3}{(3+10t)} diag\left(1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2\right)$ a derivation of $\mathfrak{n}_2$ for every t.

Remark 4.4 The limit can be also computed fixing the G_2-structure and changing the Lie bracket as in [23]. We evolve the Lie brackets $\mu(t)$ instead of the 3-form defining the G_2-structure and we can show that the corresponding bracket flow has a solution for every t. Indeed, if we fix on $\mathbb{R}^7$ the 3-form $x^{147}+x^{267}+x^{357}+x^{123}+x^{156}+x^{245}-x^{346}$, the basis $(x_1(t), \ldots, x_7(t))$ defines for every positive t a nilpotent Lie algebra with bracket $\mu(t)$ such that $\mu(0)$ is the Lie bracket of $\mathfrak{n}_2$. Moreover, the solution converges to the null bracket corresponding to the abelian Lie algebra.

In order to prove long time existence of solution for the Laplacian flow (18) of the closed G_2 form φ_4 on N_4, we need to study the (nonlinear) system of ordinary differential equations

$$\begin{cases} u' = +\frac{2}{3}\,\frac{2-u^3}{u^3v^3}, \\ v' = -\frac{2}{3}\,\frac{1-2u^3}{u^4v^2}, \end{cases} \tag{29}$$

with initial conditions

$$u(0) = v(0) = 1, \tag{30}$$

where $u = u(t)$ and $v = v(t)$ are differentiable real functions such that are both positive. Note that the first equation of (29) implies that $u' > 0$ since $u(0) = 1$, $u = u(t) > 0$ and $v = v(t) > 0$. Moreover, we note also that the functions at the second member of (29) are C^∞ in the domain

$$\Omega = \left\{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2^{1/3},\ v > 0\right\},$$

in the phase plane. Then, for every point $(u_0, v_0) \in \Omega$, there exists a unique maximal solution (u, v), which has (u_0, v_0) as initial condition, and with existence domain a certain open interval I such that either

$$\lim_{t \to \inf I} \left(u(t)^2 + v(t)^2\right) = +\infty,$$

or

$$\lim_{t \to \inf I} \left(u(t), v(t)\right) \in \partial\Omega,$$

and either

$$\lim_{t \to \sup I} \left(u(t)^2 + v(t)^2 \right) = +\infty,$$

or

$$\lim_{t \to \sup I} \left(u(t), v(t) \right) \in \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω.

Proposition 4.5 *The maximal solution* $\left(u(t), v(t)\right)$ *of* (29)*, satisfying the initial conditions* (30)*, belongs to the trajectory of equation*

$$v = \frac{1}{\sqrt{u(2-u^3)}}. \tag{31}$$

Proof From (29) we obtain

$$\frac{dv}{du} = -\frac{v(1-2u^3)}{u(2-u^3)},$$

that is,

$$\frac{dv}{v} = -\frac{1-2u^3}{u(2-u^3)}\, du.$$

Integrating this equation and using (30), we have

$$\log v = \log\left(u(2-u^3)^{-1/2} \right).$$

Therefore,

$$v = \frac{1}{\sqrt{u(2-u^3)}}.$$

□

As a consequence we have the following corollary.

Corollary 4.6 *The maximal solution of* (29)–(30)*,*

$$I \ni t \mapsto \left(u(t), v(t) \right) \in \Omega$$

parameterizes the whole curve (31)*. Moreover, the maximal solution is defined in the interval*

$$I = (t_{min}, +\infty),$$

where

$$t_{min} = -\frac{3}{2} \int_0^1 \frac{x^{3/2}}{(2-x^3)^{5/2}} dx, \tag{32}$$

and

$$\begin{cases} \lim_{t\to t_{min}} u(t) = 0, \\ \lim_{t\to t_{min}} v(t) = +\infty, \end{cases} \qquad \begin{cases} \lim_{t\to +\infty} u(t) = 2^{1/3}, \\ \lim_{t\to +\infty} v(t) = +\infty. \end{cases}$$

Proof Let $I = (t_{min}, t_{max})$ the existence interval of the maximal solution $(u(t), v(t))$ of (29) satisfying the initial conditions (30). Using the previous proposition and the first equation of (29) we see that

$$v(t) = (2u(t) - u(t)^4)^{-1/2}, \qquad u'(t) = -\frac{2u(t)^3 - 4}{3u(t)^3 v(t)^3},$$

which imply

$$u'(t) = \frac{2}{3}\frac{(2 - u(t)^3)^{\frac{5}{2}}}{u(t)^{\frac{3}{2}}}.$$

We define the functions $x(t)$ and $f(x)$ by

$$x(t) = u(t), \quad f(x) = \frac{2}{3}\frac{(2 - x^3)^{\frac{5}{2}}}{x^{\frac{3}{2}}}.$$

In order to find t_{max}, we can use that $\frac{dx}{dt} = f(x(t))$ or, equivalently,

$$\frac{dx}{f(x)} = dt.$$

So, in particular, we have

$$\frac{dt}{dx} = \frac{3}{2}x^{\frac{3}{2}}(2 - x^3)^{-\frac{5}{2}}.$$

Note that the function $\frac{3}{2}x^{\frac{3}{2}}(2 - x^3)^{-\frac{5}{2}}$ is increasing from 0, for $x = 0$, to $+\infty$, for $x = 2^{\frac{1}{3}}$. Then, integrating $\frac{dx}{f(x)} = dt$ between t_{min} and 0, and using that $x(t_{min}) = 0$ and $x(0) = 1$, we have that t_{min} is finite and equal to the real number

$$t_{min} = -\frac{3}{2}\int_0^1 x^{3/2}(2 - x^3)^{-5/2} dx.$$

Similarly, in order to find t_{max} we integrate again $\frac{dx}{f(x)} = dt$ between 0 and t_{max}. Since $x(t_{max}) = 2^{\frac{1}{3}}$ we get

$$t_{max} = -\frac{3}{2}\int_1^{2^{\frac{1}{3}}} x^{\frac{3}{2}}(2 - x^3)^{-\frac{5}{2}} dx,$$

which implies that t_{max} is $+\infty$ because this integral is not defined in $x = 2^{\frac{1}{3}}$. □

Theorem 4.7 *There exists a solution $\varphi_4(t)$ of the Laplacian flow of φ_4 on N_4 defined in the interval $I = (t_{min}, +\infty)$, where t_{min} is the negative real number given by the elliptic integral*

$$t_{min} = -\frac{3}{2}\int_0^1 x^{3/2}(2-x^3)^{-5/2}dx.$$

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in N_4, as t goes to infinity.

Proof Let us consider some differentiable real functions $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ and $h_j = h_j(t)$ $(j = 1, 2, 3)$ depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$, $h_j(0) = 0$ and $f_i(t) \neq 0$, for any $t \in I$ and for any i and j. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_4 defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 4, \quad x^5 = x^5(t) = f_5(t)e^5 + h_1(t)e^1,$$
$$x^6 = x^6(t) = f_6(t)e^6 + h_2(t)e^2, \quad x^7 = x^7(t) = f_7(t)e^7 + h_3(t)e^4.$$

The structure equations of N_4 with respect to this basis are

$$\begin{aligned} dx^i &= 0, \quad i = 1, 2, 4, 5, \quad dx^3 = \frac{f_3}{f_{12}}x^{12}, \\ dx^6 &= \frac{f_6}{f_{13}}x^{13} + \frac{f_6}{f_{24}}x^{24}, \quad dx^7 = \frac{f_7}{f_{15}}x^{15}. \end{aligned} \tag{33}$$

For any $t \in I$, we define the G_2 form $\varphi_4(t)$ on N_4 by

$$\begin{aligned} \varphi_4(t) &= -x^{124} - x^{456} + x^{347} + x^{135} + x^{167} + x^{257} - x^{236} \\ &= \Big(-f_{124} - f_4h_{12} - f_2h_{13} + f_1h_{23}\Big)e^{124} - f_{456}e^{456} + f_{347}e^{347} \\ &\quad + f_{135}e^{135} + f_{167}e^{167} + f_{257}e^{257} - f_{236}e^{236} + \Big(f_{46}h_1 - f_{16}h_3\Big)e^{146} \\ &\quad - \Big(f_{45}h_2 + f_{25}h_3\Big)e^{245} + \Big(-f_{27}h_1 + f_{17}h_2\Big)e^{127}. \end{aligned} \tag{34}$$

Clearly $\varphi_4(0) = \varphi_4$ since $f_i(0) = 1$ and $h_j(0) = 0$. Moreover, using (33) and (34), one can check that $d\varphi_4(t) = 0$ if and only if

$$f_{16}(t) = f_{34}(t), \quad f_{37}(t) = f_{56}(t),$$

for any t.

To study the flow (18) of φ_4, we need to determine the functions f_i, h_j and the interval I so that $\frac{d}{dt}\varphi_4(t) = \Delta_t \varphi_4(t)$, for $t \in I$. On the one hand, using (34) we have

$$\begin{aligned} \frac{d}{dt}\varphi_4(t) = & \Big(- f_{124} - f_4 h_{12} - f_2 h_{13} + f_1 h_{23}\Big)' e^{124} - \Big(f_{456}\Big)' e^{456} + \Big(f_{347}\Big)' e^{347} \\ & + \Big(f_{135}\Big)' e^{135} + \Big(f_{167}\Big)' e^{167} + \Big(f_{257}\Big)' e^{257} - \Big(f_{236}\Big)' e^{236} \\ & + \Big(f_{46} h_1 - f_{16} h_3\Big)' e^{146} \\ & - \Big(f_{45} h_2 + f_{25} h_3\Big)' e^{245} + \Big(- f_{27} h_1 + f_{17} h_2\Big)' e^{127}. \end{aligned} \tag{35}$$

On the other hand,

$$*_t \varphi_4(t) = x^{3567} + x^{1237} + x^{1256} - x^{2467} + x^{2345} + x^{1457} + x^{1346}.$$

So, x^{3567} and x^{2467} are the nonclosed summands in $*_t \varphi_4(t)$.

Then, for $\Delta_t \varphi_4(t) = -d *_t d *_t \varphi_4(t)$ we obtain

$$\begin{aligned} \Delta_t \varphi_4(t) = & - \left(f_{124} \left(\frac{f_3^2}{f_1^2 f_2^2} + \frac{f_6^2}{f_2^2 f_4^2} \right) - \frac{f_{37} h_3}{f_{15}} - \frac{f_6^2 h_1}{f_{13}} \right) e^{124} \\ & + f_{135} \left(\frac{f_6^2}{f_1^2 f_3^2} + \frac{f_7^2}{f_1^2 f_5^2} \right) e^{135} + \frac{f_5 f_6^2}{f_{13}} e^{245} + \frac{f_3 f_7^2}{f_1 f_5} e^{127}. \end{aligned} \tag{36}$$

Comparing (35) and (36) we see that the functions f_i, h_1 and h_3 satisfy

$$f_{167}(t) = f_{236}(t) = f_{257}(t) = f_{347}(t) = f_{456}(t) = 1, \quad f_{46}(t) h_1(t) - f_{16}(t) h_3(t) = 0,$$

for any $t \in I$. But these equations are satisfied if

$$f_1 = f_{23}^2, \qquad f_4 = f_2, \qquad f_5 = f_3, \qquad f_6 = f_7 = \frac{1}{f_{23}}, \qquad h_1 = f_2 f_3^2 h_3. \tag{37}$$

Using (37), we write (35) and (36) in terms of f_i, h_1 and h_3. Then, we see that $\frac{d}{dt}\varphi_4(t) = \Delta_t \varphi_4(t)$ if and only if

$$\begin{aligned} & f_1 = u \cdot v, \quad f_2 = f_4 = v^{1/2}, \quad f_3 = f_5 = u^{1/2}, \quad f_6 = f_7 = (uv)^{-1/2}, \\ & h_1 = \tfrac{1}{2} u^{5/2} v - \tfrac{1}{2} u^{1/2}, \quad h_2 = 0, \quad h_3 = \tfrac{1}{2} u^{3/2} v^{1/2} - \tfrac{1}{2} (uv)^{-1/2}, \end{aligned} \tag{38}$$

where $u = u(t)$ and $v = v(t)$ are differentiable real functions satisfying the system of ordinary differential equations (29) with initial conditions (30). By Corollary 4.6, we know that the system (29)–(30) has a solution $u = u(t), v = v(t)$ defined in $I = (t_{min}, +\infty)$. Then, taking into account (34) and (38), the family of closed G_2 forms $\varphi_4(t)$ solving (18) for φ_4 is given by

$$\varphi_4(t) = \frac{1}{4}e^{124}\left(-u^4v^2 + 2u^2v - 4uv^2 - 1\right) + \frac{1}{2}e^{127}\left(u^2v - 1\right) + u^2ve^{135} + e^{167} - e^{236} + \frac{1}{2}e^{245}\left(u^2v - 1\right) + e^{257} + e^{347} - e^{456},$$

for $t \in (t_{min}, +\infty)$. The underlying metric $g(t)$ of this solution converges to a flat metric. To check that the corresponding manifold in the limit is flat, we note that all non-vanishing coefficients of the Riemannian curvature $R(t)$ of $g(t)$ are proportional to the function $2u(t) - u^4(t)$. According with Corollary 4.6), we have that the function $u(t)$ satisfies

$$\lim_{t \to +\infty} u(t) = 2^{1/3},$$

and so

$$\lim_{t \to +\infty} R(t) = 0.$$

□

Concerning the Laplacian flow (18) of the closed G_2 form φ_6 on N_6 we have the following.

Theorem 4.8 *The Laplacian flow of φ_6 has a solution $\varphi_6(t)$ on N_6 defined in the interval $I = (t_{min}, +\infty)$, where t_{min} is the negative real number given by (32). Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in N_6, as t goes to infinity.*

Proof We take differentiable real functions $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ and $h_j = h_j(t)$ $(j = 1, 2)$ depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$, $h_j(0) = 0$ and $f_i(t) \neq 0$, for any $t \in I$ and for any i and j. Now, for each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_6 defined by

$$\begin{aligned} x^i &= x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 5, \\ x^6 &= x^6(t) = f_6(t)e^6 + h_1(t)e^2, \\ x^7 &= x^7(t) = f_7(t)e^7 + h_2(t)e^3. \end{aligned}$$

For any $t \in I$, let $\varphi_6(t)$ the G_2 form on N_6 defined by

$$\varphi_6(t) = x^{123} + x^{145} + x^{167} + x^{257} - x^{246} + x^{347} + x^{356}. \tag{39}$$

In order to study the flow (18) of φ_6, we proceed as in the proof of Theorem 4.7. We see that the forms $\varphi_6(t)$ defined by (39) are a solution of (18) if and only if the functions f_i, h_1 and h_2 satisfy

$$f_1 = u \cdot v, \qquad f_2 = f_3 = v^{1/2}, \qquad f_4 = f_5 = u^{1/2},$$
$$f_6 = f_7 = (uv)^{-1/2}, \qquad h_1 = h_2 = -\frac{1}{2}(uv)^{-1/2} + \frac{1}{2}u^{3/2}v^{1/2},$$

where $u = u(t)$ and $v = v(t)$ are differentiable real functions satisfying the system of ordinary differential equations

$$\begin{cases} u' = \dfrac{2}{3}\,\dfrac{2 - u^3}{u^3 v^3}, \\ v' = -\dfrac{2}{3}\,\dfrac{1 - 2u^3}{u^4 v^2}, \end{cases} \tag{40}$$

with initial conditions

$$u(0) = v(0) = 1. \tag{41}$$

Clearly, the systems (40)–(41) and (29)–(30) are the same. Thus, the maximal solution of (40)–(41) satisfies the properties expressed in Corollary 4.6 for the maximal solution of (29)–(30).

To finish the proof we see that, for $t \in (t_{min}, +\infty)$, the expression of $\varphi_6(t)$ is given by

$$\varphi_6(t) = \frac{1}{4}\Big(1 + 4uv^2 - 2u^2v + u^4v^2\Big)e^{123} + e^{347} + e^{356} + e^{167} - e^{246} + e^{257}$$
$$+ u^2ve^{145} + \frac{1}{2}\Big(1 - u^2v\Big)\Big(e^{136} - e^{127}\Big).$$

The underlying metric $g(t)$ of this solution converges to a flat metric. To check that the limit metric is flat, we note that all non-vanishing coefficients of the Riemannian curvature $R(t)$ of $g(t)$ are proportional to the function

$$u^p(t)(2 - u^3(t))^q,$$

where p and q are real numbers satisfying that $q > 0$. According with Corollary 4.6), we have that the function $u(t)$ satisfies

$$\lim_{t\to+\infty} u(t) = 2^{1/3},$$

and so

$$\lim_{t\to+\infty} R(t) = 0.$$

□

Remark 4.9 Note that surprising in the N_4 and N_6 cases we get the same system of equations.

Finally, for the Laplacian flow of the closed G_2 form φ_{12} on N_{12} we have the following.

Theorem 4.10 *The family of closed G_2 forms $\varphi_{12}(t)$ on N_{12} given by*

$$\varphi_{12}(t) = -e^{124} + e^{167} + f(t)^6 e^{135} - f(t)^6 e^{236} + e^{257} + e^{347} - e^{456}, \ t \in (-3, +\infty) \tag{42}$$

is the solution of the Laplacian flow of φ_{12}, where $f = f(t)$ is the function

$$f(t) = \left(\frac{1}{3}t + 1\right)^{1/8}.$$

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in N_{12}, as t goes to infinity.

Proof Let $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is an open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_{12} defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 7.$$

Then, from (16) the structure equations of N_{12} with respect to this basis are

$$\begin{aligned}
&dx^i = 0, \quad i = 1, 2, 3, && dx^4 = \frac{\sqrt{3}}{6}\frac{f_4}{f_{12}}x^{12},\\
&dx^5 = -\frac{1}{4}\frac{f_5}{f_{23}}x^{23} + \frac{\sqrt{3}}{12}\frac{f_5}{f_{13}}x^{13}, && dx^6 = -\frac{\sqrt{3}}{12}\frac{f_6}{f_{23}}x^{23} - \frac{1}{4}\frac{f_6}{f_{13}}x^{13},\\
&dx^7 = -\frac{\sqrt{3}}{6}\frac{f_7}{f_{34}}x^{34} + \frac{\sqrt{3}}{12}\frac{f_7}{f_{25}}x^{25} + \frac{1}{4}\frac{f_7}{f_{26}}x^{26} + \frac{\sqrt{3}}{12}\frac{f_7}{f_{16}}x^{16} - \frac{1}{4}\frac{f_7}{f_{15}}x^{15}.
\end{aligned} \tag{43}$$

Now, for any $t \in I$, we consider the G_2 form $\varphi_{12}(t)$ on N_{12} given by

$$\begin{aligned}
\varphi_{12}(t) &= -x^{124} + x^{167} + x^{135} - x^{236} + x^{257} + x^{347} - x^{456} =\\
&= -f_{124}e^{124} + f_{167}e^{167} + f_{135}e^{135} - f_{236}e^{236} + f_{257}e^{257} + f_{347}e^{347} - f_{456}e^{456}.
\end{aligned} \tag{44}$$

Note that $\varphi_{12}(0) = \varphi_{12}$ and, for any t, the 3-form $\varphi_{12}(t)$ on N_{12} determines the metric g_t such that the basis $\{x_i = \frac{1}{f_1}e_i; i = 1, \ldots, 7\}$ of $\mathfrak{n}_{12}$ is orthonormal. So, $g_t(e_i, e_i) = f_i^2$.

We need to determine the functions f_i and the interval I so that $\frac{d}{dt}\varphi_{12}(t) = \Delta_t\varphi_{12}(t)$, for $t \in I$. Using (44) we have

$$\begin{aligned}
\frac{d}{dt}\varphi_{12}(t) = &-(f_{124})'e^{124} + (f_{167})'e^{167} + (f_{135})'e^{135} - (f_{236})'e^{236} +\\
&+ (f_{257})'e^{257} + (f_{347})'e^{347} - (f_{456})'e^{456}.
\end{aligned} \tag{45}$$

Now, we calculate $\Delta_t\varphi_{12}(t) = -d *_t d *_t \varphi_{12}(t)$. On the one hand, we have

$$*_t\, \varphi_{12}(t) = x^{3567} - x^{2467} + x^{2345} + x^{1457} + x^{1346} + x^{1256} + x^{1237}. \tag{46}$$

So, x^{2467} and x^{1457} are the unique non closed summands in $*_t\varphi_{12}(t)$. Then, taking into account the structure equations (43) and that $x^i(t) = f_i(t)e^i$, $1 \leq i \leq 7$ we obtain

$$\begin{aligned}\Delta_t\varphi_{12}(t) = &-\frac{(f_{15}+f_{26})(f_5^2f_6^2+f_3^2f_7^2)}{16 f_1f_2f_3f_5f_6}(e^{236}-e^{135})+\\ &+\frac{(f_{15}+f_{26})(f_5^2f_6^2-f_3^2f_7^2)}{16\sqrt{3} f_1f_2f_3f_5f_6}(e^{136}+e^{235}).\end{aligned} \tag{47}$$

Comparing (45) and (47), in particular, we have that

$$(f_{124})' = (f_{167})' = (f_{257})' = (f_{347})' = (f_{456})' = 0,$$

and since $\varphi_{12}(0) = \varphi_{12}$ this imply that

$$f_{124}(t) = f_{167}(t) = f_{257}(t) = f_{347}(t) = f_{456}(t) = 1, \tag{48}$$

for any $t \in I$. From the equation (48) we obtain that

$$f_1 = f_1;\quad f_2 = f_2;\quad f_3 = (f_1f_2)^2;\quad f_4 = \frac{1}{f_1f_2};\quad f_5 = f_1;\quad f_6 = f_2;\quad f_7 = \frac{1}{f_1f_2}.$$

Let us consider $f = f_1 = f_2$. With these concrete values (45) and (47) become

$$\frac{d}{dt}\varphi_{12}(t) = (f^6(t))'(e^{135} - e^{236}), \tag{49}$$

and

$$\Delta_t\varphi_{12}(t) = \frac{f(t)^{-2}}{4}\left(e^{135} - e^{236}\right), \tag{50}$$

respectively. From (49) and (50) finding a solution of the Laplacian flow is equivalent to solve $f^7f' = \frac{1}{24}$. Integrating this equation, we obtain

$$f^8 = \frac{1}{3}t + B, \quad B = constant.$$

But $\varphi(0) = \varphi_{12}$ implies that $f(0) = 1$, that is, $B = 1$. Hence

$$f(t) = \left(\frac{1}{3}t + 1\right)^{1/8},$$

and so the one-parameter family of 3-forms $\{\varphi_{12}(t)\}$ given by (42) is the solution of the Laplacian flow of φ_{12} on N_{12}, and it is defined for every $t \in (-3, +\infty)$.

Finally, we study the behavior of the underlying metric $g(t)$ of such a solution in the limit. If we think of the Laplacian flow as a one-parameter family of G_2 manifolds with a closed G_2-structure, it can also be checked that, in the limit, the resulting manifold has vanishing curvature. Denote by $g(t)$, $t \in (-3, +\infty)$, the metric on N_{12} induced by the G_2 form $\varphi_{12}(t)$ defined by (42). Then, $g(t)$ has the following expression

$$
\begin{aligned}
g(t) = & \left(\frac{1}{3}t+1\right)^{1/4}(e^1)^2 + \left(\frac{1}{3}t+1\right)^{1/4}(e^2)^2 + \left(\frac{1}{3}t+1\right)^{-1}(e^3)^2 \\
& + \left(\frac{1}{3}t+1\right)^{-1/2}(e^4)^2 + \left(\frac{1}{3}t+1\right)^{1/4}(e^5)^2 + \left(\frac{1}{3}t+1\right)^{1/4}(e^6)^2 \\
& + \left(\frac{1}{3}t+1\right)^{-1/2}(e^7)^2.
\end{aligned}
$$

Concretely, every non vanishing coefficient appearing in the expression of the Riemannian curvature $R(t)$ of $g(t)$ is proportional to $(t+3)^{-1}$. Therefore, $\lim_{t\to+\infty} R(t) = 0$. □

Remark 4.11 Note that, for every $t \in (-3, +\infty)$, the metric $g(t)$ is a nilsoliton on the Lie algebra $\mathfrak{n}_{12}$ of N_{12}. In fact, with respect to the orthonormal basis $(x_1(t), \ldots, x_7(t))$, we have

$$
Ric(g(t)) = -\frac{3}{4(3+t)}Id + \frac{3}{8(3+t)}diag(1,1,1,2,2,2,3) = \frac{3}{(3+t)}Ric(g(0))
$$

with $\frac{3}{8(3+t)}diag(1,1,1,2,2,2,3)$ a derivation of $\mathfrak{n}_{12}$ for every t.

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References

1. Alekseevskii, D., Kimelfeld, B.: Structure of homogeneous Riemannian spaces with zero Ricci curvature. Funktional Anal. i Prilozen **9**, 5–11 (1975). English translation: Functional. Anal. Appl. **9**, 97–102 (1975)
2. Bryant, R.L.: Metrics with exceptional holonomy. Ann. Math. **126**, 525–576 (1987)
3. Bryant, R.L.: Some remarks on G_2 structures. In: Proceedings of Gökova Geometry-Topology Conference 2005, pp. 75–109, Gökova Geometry/Topology Conference (GGT), Gökova (2006)
4. Bryant, R.L., Xu, F.: Laplacian flow for closed G_2-structures: short time behavior. arXiv:1101.2004 [math.DG]
5. Cao, H.D.: Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds. Invent. Math. **81**, 359–372 (1985)
6. Cleyton, R., Ivanov, S.: Curvature decomposition of G_2-manifolds. J. Geom. Phys. **58**, 1429–1449 (2008)
7. Chow, B., Chu, S.-C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci flow: techniques and applications, part I: geometric aspects. AMS Math. Surv. Mon. **135** (2007). Am. Math. Soc., Providence

8. Conti, D., Fernández, M.: Nilmanifolds with a calibrated G_2-structure. Differ. Geom. Appl. **29**, 493–506 (2011)
9. Fernández-Culma, E.A.: Classification of 7-dimensional Einstein Nilradicals. Transform. Groups **17**, 639–656 (2012)
10. Fernández-Culma, E.A.: Classification of 7-dimensional Einstein Nilradicals II. arXiv:1105.4493 [math.DG]
11. Fernández-Culma, E.A.: Classification of Nilsoliton metrics in dimension seven. J. Geom. Phys. **86**, 164–179 (2014)
12. Fernández, M., Fino, A., Manero, V.: G_2-structures on Einstein solvmanifolds. Asian J. Math. **19**, 321–342 (2015)
13. Fernández, M., Gray, A.: Riemannian manifolds with structure group G_2. Ann. Mat. Pura Appl. **132**, 19–45 (1982)
14. Hamilton, R.S.: The Ricci flow on surfaces. Contemp. Math. **71**, 237–261 (1988)
15. Heber, J.: Noncompact homogeneous Einstein spaces. Invent. Math. **133**, 279–352 (1998)
16. Ivey, T.: Ricci solitons on compact three-manifolds. Differ. Geom. Appl. **3**, 301–307 (1993)
17. Jablonski, M.: Homogeneous Ricci solitons. J. Reine Angew. Math. **699**, 159–182 (2015)
18. Kadioglu, H., Payne, T.L.: Computational methods for nilsoliton metric Lie algebras I. J. Symb. Comput. **50**, 350–373 (2013)
19. Lauret, J.: Ricci soliton homogeneous nilmanifolds. Math. Ann. **319**(4), 715–733 (2001)
20. Lauret, J.: Einstein solvmanifolds and nilsolitons. In: New Developments in Lie Theory and Geometry, Contemp. Math. vol. 491, Amer. Math. Soc, Providence, RI, pp. 1–35 (2009)
21. Lauret, J.: Einstein solvmanifolds are standard. Ann. Math. **172**, 1859–1877 (2010)
22. Lauret, J.: Ricci soliton solvmanifolds. J. Reine Angew. Math. **650**, 1–21 (2011)
23. Lauret, J.: The Ricci flow for simply connected nilmanifolds. Commun. Anal. Geom. **19**(5), 831–854 (2011)
24. Milnor, J.: Curvatures of left invariant metrics on Lie groups. Adv. Math. **21**, 293–329 (1976)
25. Naber, A.: Noncompact shrinking four solitons with nonnegative curvature. J. Reine Angew. Math. **645**, 125–153 (2010)
26. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:0211159 [math.DG]
27. Petersen, P., Wylie, W.: On gradient Ricci solitons with symmetry. Proc. Am. Math. Soc. **137**, 2085–2092 (2009)
28. Schulte-Hengesbach, F.: Half-flat structures on Lie groups. PhD Thesis, Hamburg (2010). http://www.math.uni-hamburg.de/home/schulte-hengesbach/diss
29. Xu, F., Ye, R.: Existence, convergence and limit map of the Laplacian flow. arXiv:0912.0074 [math.DG]