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Solutions of the Laplacian flow and coflow of a locally conformal parallel G_2-structure

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Abstract. We study the Laplacian flow of a G_2-structure where this latter structure is claimed to be locally conformal parallel. The first examples of long time solutions of this flow with the locally conformal parallel condition are given. All of the solutions are ancient and Laplacian soliton of shrinking type. These examples are one-parameter families of locally conformal parallel G_2-structures on rank-one solvable extensions of six-dimensional nilpotent Lie groups. The found solutions are used to construct long time solutions to the Laplacian coflow starting from a locally conformal parallel structure. We also study the behavior of the curvature of the solutions obtaining that for one of the examples the induced metric is Einstein along all the flow (resp. coflow).

Contents

Introduction

A G_2-structure on a 7-dimensional smooth manifold M is a reduction to the exceptional Lie group G_2 of the structure group $GL(7, \mathbb{R})$ of the frame bundle of M. We call G_2-*manifold* a 7-dimensional manifold endowed with a G_2-structure. The presence of a G_2-structure is equivalent to the existence of a globally defined 3-form

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φ, which is called the G_2 *form* or the *fundamental* 3-*form* and it can be described locally as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \qquad (1)$$

with respect to some local basis $\{e^1, \ldots, e^7\}$ of the 1-forms on M, which we call an *adapted basis*. The notation $e^{i_1 \cdots i_k}$ stands for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. The fundamental 3-form φ is stable in the sense that its orbit at each point $p \in M$ under the natural action of the group $GL(T_pM)$ is open (see [13]).

The existence of a G_2 form φ on a manifold M induces a Riemannian metric g_φ and a volume element vol_φ on M related by the formula:

$$g_\varphi(X, Y) vol_\varphi = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi, \qquad (2)$$

for any vector fields X, Y on M.

If ∇ denotes the Levi–Civita connection with respect to the induced metric g_φ, Fernández and Gray [7] defined many different G_2-structures in terms of the intrinsic torsion of the G_2-structure given by $\nabla\varphi$. Moreover, it is proved that the intrinsic torsion is completely determined by the exterior derivative of the G_2 form φ and the 4-form $*\varphi$, where $*$ denotes the Hodge star operator induced by the metric and the volume form (2). The most restrictive class of G_2-structures is the one contaning the so called parallel G_2-structures, which are covariantly constant with respect to ∇. Manifolds endowed with such structure are characterized by the condition that both φ and $*\varphi$ are closed. Equivalently, the G_2-form φ and the Riemannian holonomy group of the underlying metric g_φ is a subgroup of G_2 being in addition Ricci-flat [2].

The development of flows in Riemannian geometry has been mainly motivated by the study of the Ricci flow. The same techniques are also useful in the study of flows involving other geometrical structures, like for example, the Kähler Ricci flow.

Given a closed (or calibrated in the terminology of Harvey and Lawson [12]) G_2-structure φ_0 on a manifold M, that is $d\varphi_0 = 0$, Bryant introduced in [3] a natural flow, the so-called *Laplacian flow*, given by the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi_0, \end{cases}$$

where Δ_t is the Hodge Laplacian operator of the metric $g_{\varphi(t)}$ determined by $\varphi(t)$. The short time existence and uniqueness of solution for the Laplacian flow of any closed G_2-structure, on a compact manifold M, has been proved by Bryant and Xu in the unpublished paper [4]. Also, long time existence and convergence of the Laplacian flow starting near a torsion-free G_2-structure was proved in the unpublished paper [21] whenever the torsion of φ is sufficiently small. In the last years, Lotay and Wei [16–18] have obtained many results concerning the properties of the Laplacian flow.

In [6] the first examples of noncompact manifolds with long time existence of the solution for the Laplacian flow of a closed G_2-structure are shown. Those examples are nilpotent Lie groups admitting an invariant closed G_2-structure which determines the nilsoliton metric. Recently in [9] the authors studied the Laplacian flow of a closed G_2-structure on warped products of the form $M \times S^1$ where the base space is a 6-dimensional compact manifold endowed with an SU(3)-structure. Impossing the warping function to be constant they find sufficient conditions for the existence of solution of the Laplacian flow and present some examples where M is a six-dimensional solvmanifold. In a similar way, in [19] the authors find solutions for the Laplacian coflow on warped products $M \times S^1$.

Karigiannis et al. [14] introduced the *Laplacian coflow* (or *coflow* for short). In this case the initial G_2-form φ_0 is claimed to be coclosed (or cocalibrated as in [12]), i.e. $d\psi_0 = 0$, where $\psi_0 = *\varphi_0$. The equations of this flow are given by

$$\begin{cases} \frac{d}{dt}\psi(t) = -\Delta_t \psi(t), \\ d\psi(t) = 0, \\ \psi(0) = \psi_0, \end{cases}$$

with $\psi(t) = *_t\varphi(t)$ the Hodge dual 4-form of the G_2-structure $\varphi(t)$ and Δ_t is the Hodge Laplacian operator with respect to the metric $g_{\varphi(t)}$ induced by $\varphi(t)$. Unlike the Laplacian flow, up to now short time existence of solution of the coflow is not known. Assuming short time existence and uniqueness of solution, it is shown in [14] that the condition of the initial G_2-form φ_0 to be coclosed (equiv. ψ_0 closed) is preserved along the flow.

In [10] Grigorian introduced a modified version of the Laplacian coflow which is called the *modified Laplacian coflow* and proved short time existence and uniqueness of solution for this modified flow. Recently in [1] explicit solutions for the coflow and the modified Laplacian coflow have been described. These solutions are one-parameter families of G_2-structures defined on the 7-dimensional Heisenberg Lie group. The solutions for the coflow are always ancient, i.e., defined for all time $-\infty < t < T$, with $T < \infty$, for every initial cocalibrated G_2-structure. The condition of the induced metric to be Ricci soliton is preserved along the coflow. For overviews on these topics, see [11,15].

In this paper we are concerned with studying the Laplacian flow, resp. coflow, on locally conformal parallel G_2-structures, (LCP for short) as they play in some sense an intermediate role between closed and coclosed G_2-structures. LCP G_2-structures are characterized by the fact that at each point $p \in M$, there is some differentiable function f defined on a local neighbourhood of p such that the underlying metric g_φ can be modified locally to a metric $\widetilde{g}$ with holonomy a subgroup of G_2 by means of a conformal change $\widetilde{g} = e^{2f} g_\varphi$. Equivalently, the LCP condition is given in terms of the exterior derivatives of φ and $*\varphi$ by:

$$d\varphi = 3\,\tau \wedge \varphi, \quad d * \varphi = 4\,\tau \wedge * \varphi, \tag{3}$$

with τ the Lee 1-form. These G_2-structures are of type $\mathcal{X}_4$ in the sense of Fernández-Gray, see [7].

In order to describe the first examples of solution of these flows we will consider the class of solvable Lie groups described by Fino and Chiossi [5] constructed as rank-one solvable extensions of nilpotent Lie groups admitting left-invariant Locally Conformal Parallel G_2-structures.

The paper is structured as follows: in Sect. 1, we review some explicit examples on Lie groups solving the Laplacian flow and the Laplacian coflow. This allows us to set a generic ansatz for solving flows related with G_2-structures on Lie groups which will be useful in the rest of the paper. Section 2 starts by introducing rank-one solvable extensions of nilpotent Lie groups. We detailed in Proposition 2.1 the list of Lie algebras found in [5, Theorem 1] underlying the seven dimensional solvable Lie groups constructed in this way and admitting a left-invariant LCP G_2-structure. The rest of this section deals with exploring the Laplacian flow under the assumption of solutions defined in Sect. 1 either setting necessary and sufficient conditions preserving the LCP condition or describing the $\Delta_t\varphi(t)$ in a suitable form. Sections 3 and 4 are devoted to construct explicit examples of solutions to the Laplacian flow and coflow preserving the LCP-condition. In Theorem 3.1 we present an explicit solution for the Laplacian flow where the LCP-condition is preserved, notice that the metric induced by the solution remains Einstein along the flow. The rest of Sect. 3 is devoted to obtain solutions for the remaining solvable Lie groups described by Chiossi and Fino. The solutions of the flow turn out to be Laplacian solitons. In Sect. 4 we obtain relations between the sets of solutions of the Laplacian flow and coflow where the LCP-condition is preserved (see Theorem 4.1). Finally, in the "Appendix" we include the expressions of the curvature for the metric induced by the solutions of the Laplacian flow previously obtained.

1. Laplacian flows on Lie groups

In the last years there has been a wide interest in finding solutions for the Laplacian flow and related notions have been explored such as new examples with extra properties. In general, flows of G_2-structures are of the form

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t\varphi(t), \\ \varphi(t) \in \mathcal{C}, \end{cases} \tag{4}$$

where Δ_t denotes the corresponding Hodge Laplacian operator, $\mathcal{C}$ is a specific class of G_2-structures and t lives in an open real interval. We will refer to it as *$\mathcal{C}$-flow*.

The first author considering flows of G_2-structures was Bryant [3]. The objective of considering flows of G_2-structures was to obtain examples of G_2-manifolds without torsion as the result of certain evolution of other G_2-structures with torsion. Thus, Bryant considered the so-called *Laplacian flow* of a G_2-structure φ_0 which is given by (4) where $\varphi(t)$ is supposed to be closed. On compact manifolds short time existence and uniqueness of solution for the Laplacian flow of a closed G_2-structure has been proved by Bryant and Xu [4]. Xu and Ye [21] proved long time existence and convergence of solution of the Laplacian flow starting near a torsion-free G_2-structure. In the last years Lotay and Wei in the series of papers [16–18] have obtained important results concerning long time existence and convergence of solutions of the Laplacian flow.

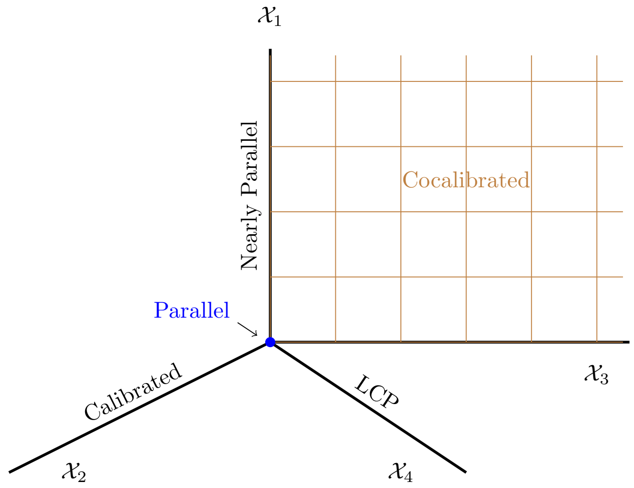

Fig. 1. Principal classes of G_2-structures

On the other hand, in [14] Karigiannis et al. introduced the *Laplacian coflow*. This latter flow can be considered as the analogue to the Laplacian flow in which the fundamental 3-form is claimed to be coclosed instead of closed. Thus, this flow is given by the equations

$$\begin{cases} \frac{d}{dt}\psi(t) = -\Delta_t \psi(t), \\ \psi(0) = \psi_0, \\ d\psi(t) = 0, \end{cases} \tag{5}$$

with $\psi(t) = *_t\varphi(t)$ and $*_t$ denoting the Hodge star operator. As far as the authors know, short time existence and uniqueness of solution for this latter flow is not known. In [10] Grigorian introduced a modified version of this flow called modified Laplacian coflow for which he proved short time existence and uniqueness of solution.

1.1. Torsion of G_2*-structures*

The torsion of a G_2-structure can be identified with the covariant derivative of the fundamental form φ with respect to the Levi–Civita connection of the induced metric. As it is described in [7], it can be decomposed into four G_2 irreducible components, namely X_1, X_2, X_3 and X_4. Thus, a G_2-structure is said to be of type $\mathcal{P}$, $\mathcal{X}_i$, $\mathcal{X}_i \oplus \mathcal{X}_j$, $\mathcal{X}_i \oplus \mathcal{X}_j \oplus \mathcal{X}_k$ or $\mathcal{X}$ if $\nabla\varphi$ lies in $\{0\}$, X_i, $X_i \oplus X_j$, $X_i \oplus X_j \oplus X_k$ or $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$, respectively. Hence, there exist 16 different classes of G_2-structures. Some of the principal classes are summarized in Table 1 and Fig. 1.

Equivalently these classes of G_2-structures can be characterized in terms of the exterior derivatives of φ and $*\varphi$ [7].

Table 1. Some classes of G_2-structures

Class	Type	Exterior derivatives
$\mathcal{P}$	Parallel	$d\varphi = 0, \quad d * \varphi = 0$
$\mathcal{X}_2$	Calibrated (or closed)	$d\varphi = 0$
$\mathcal{X}_4$	Locally conformal parallel (LCP)	$d\varphi = 3\,\tau \wedge \varphi, \quad d * \varphi = 4\,\tau \wedge * \varphi$
$\mathcal{X}_1 \oplus \mathcal{X}_3$	Cocalibrated (or coclosed)	$d * \varphi = 0$

1.2. Examples of solution

The first examples of long time solutions for the Laplacian flow of closed G_2-structures ($\mathcal{C} = \mathcal{X}_2$) were described in [6] using nilpotent Lie groups endowed with a one parameter family of left-invariant closed G_2-structures.

Example 1.1. Consider the connected and simply connected Lie group G whose underlying Lie algebra has the structure equations:

$$de^5 = e^1 \wedge e^2, \quad de^6 = e^1 \wedge e^3, \quad \text{and} \quad de^i = 0 \quad \text{for all} \quad i = 1, 2, 3, 4, 7.$$

The family of closed G_2 forms $\varphi(t)$ on G given by

$$\varphi(t) = e^{147} + e^{267} + e^{357} + f(t)^3 e^{123} + e^{156} + e^{245} - e^{346}, \qquad t \in \left(-\frac{3}{10}, +\infty\right),$$

where $f(t)$ is the positive function

$$f(t) = \left(\frac{10}{3}t + 1\right)^{\frac{1}{5}}.$$

is the solution of the Laplacian flow with initial value

$$\varphi_0 = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346}.$$

Analogously in [1] have been given explicit long time solutions for the Laplacian coflow (5).

Example 1.2. Consider the 7-dimensional Heisenberg Lie group H_7, whose corresponding Lie algebra, namely $\mathfrak{h}_7$, is given by the structure equations

$$de^7 = \frac{\sqrt{6}}{6}\left(e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6\right), \quad \text{and } de^i = 0 \quad \text{for all } i = 1, \ldots, 6.$$

The solution of the Laplacian coflow on H_7 with the initial coclosed G_2 form,

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

is given by

$$\varphi(t) = \frac{1}{f(t)}\left(e^{127} + e^{347} + e^{567}\right) + f(t)^3\left(e^{135} - e^{146} - e^{236} - e^{245}\right), \quad t \in \left(-\infty, \frac{3}{5}\right)$$

where $f(t)$ is the positive function

$$f(t) = \left(1 - \frac{5}{3}t\right)^{\frac{1}{10}}.$$

1.3. Results on Lie groups

Notice that the previous examples consist on solutions of the flows on Lie groups where a very concrete ansatz has been considered. In general, let G be a simply connected solvable Lie group of dimension 7 with Lie algebra $\mathfrak{g}$. Let $\{e^1, \dots, e^7\}$ be a basis of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, and let $f_i = f_i(t)$ $(i = 1, \dots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is a real open interval. For each $t \in I$, we define the basis $\{x^1, \dots, x^7\}$ of $\mathfrak{g}^*$ by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 7.$$

We consider the one-parameter family of left-invariant G_2-structures $\varphi(t)$ on G given by

$$\begin{aligned}\varphi(t) &= x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245} \\ &= f_{127}e^{127} + f_{347}e^{347} + f_{567}e^{567} + f_{135}e^{135} - f_{146}e^{146} - f_{236}e^{236} - f_{245}e^{245},\end{aligned} \tag{6}$$

where $f_{ijk} = f_{ijk}(t)$ stands for the product $f_i(t)f_j(t)f_k(t)$. Now, following [8] can be introduced the function $\varepsilon(i, j, k)$ on ordered indices (i, j, k) as follows:

$$\varepsilon(i, j, k) = \begin{cases} 1 & \text{if } (i, j, k) \in A = \{(1, 2, 7), (1, 3, 5), (3, 4, 7), (5, 6, 7)\}; \\ -1 & \text{if } (i, j, k) \in B = \{(1, 4, 6), (2, 3, 6), (2, 4, 5)\}. \end{cases}$$

Thus, the G_2 form φ defined in (1), can be expressed as $\varphi = \sum_{(i,j,k)\in A\cup B} \varepsilon(i, j, k)e^{ijk}$, and the family of G_2 forms $\varphi(t)$ given by (6) becomes

$$\varphi(t) = \sum_{(i,j,k)\in A\cup B} \varepsilon(i, j, k)x^{ijk}.$$

Therefore,

$$\frac{d}{dt}\varphi(t) = \sum_{(i,j,k)\in A\cup B} \varepsilon(i, j, k)\frac{df_{ijk}}{dt}e^{ijk} = \sum_{(i,j,k)\in A\cup B} \varepsilon(i, j, k)\frac{(f_{ijk})'}{f_{ijk}}x^{ijk}.$$

Moreover, we express the 3-form $\Delta_t\varphi(t)$ as a linear combination of the basis of 3-forms $\{x^{abc}\}$ as

$$\Delta_t\varphi(t) = \sum_{(i,j,k)\in A\cup B} \varepsilon(i, j, k)\Delta_{ijk}\, x^{ijk} + \sum_{1\leq i<j<k\leq 7,\ (i,j,k)\notin A\cup B} \Delta_{ijk}\, x^{ijk}, \tag{7}$$

where $\varepsilon(i, j, k)\Delta_{ijk}$ is the coefficient in x^{ijk} of $\Delta_t\varphi(t)$ if $(i, j, k) \in A \cup B$ (i.e., if $\varepsilon(i, j, k) \neq 0$), and Δ_{ijk} is the coefficient in x^{ijk} of $\Delta_t\varphi(t)$ if $(i, j, k) \notin A \cup B$. Consequently, the first equation of the $\mathcal{C}$-flow (4) (regardless of condition $\mathcal{C}$) is equivalent to the system of differential equations

$$\begin{cases} \Delta_{ijk} = \frac{(f_{ijk})'}{f_{ijk}} & \text{if } (i, j, k) \in A \cup B, \\ \Delta_{ijk} = 0 & \text{if } 1 \leq i < j < k \leq 7 \text{ and } (i, j, k) \notin A \cup B. \end{cases} \tag{8}$$

The following lemma generalizes [8, Lemma 1] and states some properties involving the Δ_{ijk} coefficients.

Lemma 1.3. *Let $\varphi(t)$ be a family of left invariant* G_2*-structures given by* (6) *on the Lie group G solving the system* (8)*. For ordered indices (i, j, k) and $(p, q, r) \in A \cup B$ and $\alpha, \beta \in \mathbb{R}$ we have*

(i) *if* $\alpha\Delta_{ijk} = \beta\Delta_{pqr}$*, then* $(f_{ijk})^{\alpha} = (f_{pqr})^{\beta}$*;*
(ii) *if* $\alpha f_{ijk}\Delta_{ijk} = \beta f_{pqr}\Delta_{pqr}$*, then* $\alpha(f_{ijk} - 1) = \beta(f_{pqr} - 1)$*.*

Proof. For (i) suppose $\alpha\Delta_{ijk} = \beta\Delta_{pqr}$. In view of (8) this is equivalent to $\alpha\frac{(f_{ijk})'}{f_{ijk}} = \beta\frac{(f_{pqr})'}{f_{pqr}}$. Notice that the last expression can be stated as $\alpha\frac{d}{dt}\ln(f_{ijk}) = \beta\frac{d}{dt}\ln(f_{pqr})$. Therefore $\frac{d}{dt}\ln\left(\frac{(f_{ijk})^{\alpha}}{(f_{pqr})^{\beta}}\right) = 0$. Hence $\ln\left(\frac{(f_{ijk})^{\alpha}}{(f_{pqr})^{\beta}}\right)$ is constant and since $f_l(0) = 1$ for all $l = 1, \ldots, 7$ we conclude that $(f_{ijk})^{\alpha} = (f_{pqr})^{\beta}$. Part (ii) is immediate. □

Notice that flows on G_2-structures whose fundamental form is claimed to be calibrated (belonging to class $\mathcal{C} = \mathcal{X}_2$) or cocalibrated (in class $\mathcal{C} = \mathcal{X}_1 \oplus \mathcal{X}_3$) have been deeply studied. Thus in view of Fig. 1 it seems natural to consider the remaining case, i.e. flows where the fundamental form is required to be locally conformal parallel ($\mathcal{C} = \mathcal{X}_4$). However, as far as the authors know, nothing has been done for flows of G_2-structures where the LCP condition is required along the flow. Therefore in this paper we are concerned with studying the Laplacian flow, resp. coflow, of an LCP G_2-structure on a manifold M, or simply the *LCP-flow*, resp. *LCP-coflow*, which can be defined as:

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = \Delta_t\varphi(t), \\ \varphi(t) \in \mathcal{X}_4 \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{d}{dt}\psi(t) = -\Delta_t\psi(t), \\ *_t\psi(t) \in \mathcal{X}_4 \end{cases} \tag{10}$$

where a G_2-structure φ belongs to class $\mathcal{X}_4$ if it satisfies Eq. (3).

2. Laplacian flow on LCP rank-one solvable extensions of nilpotent Lie groups

In this section we study the Laplacian flow on a specific set of Lie groups endowed with a left-invariant LCP G_2-structure. The associated Lie algebras of these groups are rank-one solvable extensions of 6-dimensional nilpotent Lie algebras. These solvable extensions are constructed generically as follows (see [20]). Given a n-dimensional metric nilpotent Lie algebra $(\mathfrak{n}, \langle\cdot,\cdot\rangle_{\mathfrak{n}})$, its $(n+1)$-dimensional solvable extension is a vector space $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_{n+1}$ where $e_{n+1} \notin \mathfrak{n}$ endowed with a metric $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ which is fixed on $\mathfrak{s}$ extending the one on $\mathfrak{n}$ i.e $\langle\cdot,\cdot\rangle_{\mathfrak{s}_{|\mathfrak{n}}} = \langle\cdot,\cdot\rangle_{\mathfrak{n}}$ and declaring that $\langle e_{n+1}, e_{n+1}\rangle_{\mathfrak{s}} = 1$ and $\langle e_{n+1}, \mathfrak{n}\rangle_{\mathfrak{s}} = 0$. Now, given a derivation D of the Lie algebra $\mathfrak{n}$, the Lie bracket $[\cdot,\cdot]_{\mathfrak{s}}$ on $\mathfrak{s}$ is defined as $[X, Y]_{\mathfrak{s}} = [X, Y]_{\mathfrak{n}}$ and $[e_{n+1}, Y]_{\mathfrak{s}} = DY$ for every $X, Y \in \mathfrak{n}$, i.e., $\mathrm{ad}_{e_{n+1}}|_{\mathfrak{n}} = D$.

Fino and Chiossi [5] adapt the former construction when $\mathfrak{n}$ is a six-dimensional nilpotent Lie algebra endowed with an SU(3)-structure (ω, ψ_+) and D is a derivation of $\mathfrak{n}$ being non-singular, self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ and diagonalisable by an adapted Hermitian basis $\{e_1, \dots, e_6\}$ of $\mathfrak{n}$ (the latter condition being equivalent to $(DJ)^2 = (JD)^2$). In this setting, the Maurer–Cartan equations for the seven-dimensional Lie algebra yield

$$\begin{cases} de^k = \eta_k \, e^k \wedge e^7 + \hat{d}e^k, & 1 \leq k \leq 6, \\ de^7 = 0, \end{cases}$$

where the η_k are the eigenvalues of the derivation D and $\hat{d}e^k = \sum_{1 \leq i < j \leq 6} c_{ij}^k e^{ij}$ is the exterior derivative at the 6-dimensional level $\mathfrak{n}$.

It turns out that there is a natural G_2-structure on the solvable Lie group $S = N \times \mathbb{R}$ corresponding to the 3-form:

$$\varphi = \omega \wedge e^7 + \psi_+$$

where e^7 denotes the 1-form $\langle e_7, \cdot\rangle_{\mathfrak{s}}$ and N is the nilpotent Lie group associated to the nilpotent Lie algebra $\mathfrak{n}$. Indeed, they prove that when (S, φ) is locally conformal parallel the SU(3)-structure (ω, ψ_+) is half-flat, that is, $d\omega^2 = 0$ and $d\psi_+ = 0$. More concretely, the list of Lie algebras underlying such locally conformal parallel structures in this setting is contained in the following classifying result (in a slightly different representation with respect to the original one found inside the proof of [5, Theorem 1]):

Proposition 2.1. *Let N be a nilpotent Lie group of dimension 6 endowed with an invariant SU(3)-structure (ω, ψ_+). Suppose that there is a non-singular and self-adjoint derivation D of the Lie algebra $\mathfrak{n}$ such that $(DJ)^2 = (JD)^2$. Then, on the solvable extension $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ with $ad_{e_7} = D$, the G_2-structure*

$$\varphi = \left(e^{12} + e^{34} + e^{56}\right) \wedge e^7 + e^{135} - e^{146} - e^{236} - e^{245},$$

is locally conformal parallel if and only if $\mathfrak{s}$ is isomorphic to one of the following list:

$$\mathfrak{cp}_1^m = \left(-me^{17}, -me^{27}, -me^{37}, -me^{47}, -me^{57}, -me^{67}, 0\right);$$

$$\mathfrak{cp}_2^m = \left(-\frac{4}{3}me^{17} + \frac{2}{3}me^{36}, -me^{27}, -\frac{2}{3}me^{37}, -me^{47}, -me^{57}, -\frac{2}{3}me^{67}, 0\right);$$

$$\mathfrak{cp}_3^m = \left(-\frac{3}{2}me^{17} + \frac{1}{2}me^{36} + \frac{1}{2}me^{45}, -me^{27}, -\frac{3}{4}me^{37},\right.$$
$$\left. -\frac{3}{4}me^{47}, -\frac{3}{4}me^{57}, -\frac{3}{4}me^{67}, 0\right);$$

$$\mathfrak{cp}_4^m = \left(-\frac{7}{5}me^{17} + \frac{2}{5}me^{36} + \frac{2}{5}me^{45}, -\frac{6}{5}me^{27} - \frac{2}{5}me^{46},\right.$$
$$\left. -\frac{4}{5}me^{37}, -\frac{3}{5}me^{47}, -\frac{4}{5}me^{57}, -\frac{3}{5}me^{67}, 0\right);$$

$$\mathfrak{cp}_5^m = \Big(-\frac{5}{4}me^{17} + \frac{1}{2}me^{45}, -\frac{5}{4}me^{27} - \frac{1}{2}m\,e^{46}, -me^{37},$$
$$-\frac{1}{2}me^{47}, -\frac{3}{4}me^{57}, -\frac{3}{4}me^{67}, 0\Big);$$
$$\mathfrak{cp}_6^m = \Big(-\frac{4}{3}me^{17} + \frac{1}{3}me^{36} + \frac{1}{3}me^{45}, -\frac{4}{3}me^{27} + \frac{1}{3}me^{35} - \frac{1}{3}me^{46},$$
$$-\frac{2}{3}me^{37}, -\frac{2}{3}me^{47}, -\frac{2}{3}me^{57}, -\frac{2}{3}me^{67}, 0\Big);$$
$$\mathfrak{cp}_7^m = \Big(-\frac{6}{5}me^{17} + \frac{2}{5}me^{36}, -\frac{3}{5}me^{27}, -\frac{3}{5}me^{37}, \frac{2}{5}me^{26} - \frac{6}{5}me^{47},$$
$$\frac{2}{5}me^{23} - \frac{6}{5}me^{57}, -\frac{3}{5}me^{67}, 0\Big).$$

Proof. All the Lie algebras fulfilling the hypothesis of the theorem are originally expressed (see the proof of [5, Theorem 1], expression numbers from (9) to (15)) in a basis $\{\nu^1, \dots, \nu^7\}$ of $\mathfrak{s}^*$ where the G_2-structure φ adopts the following expression:

$$\varphi = \nu^{125} - \nu^{345} + \nu^{567} + \nu^{136} + \nu^{246} - \nu^{237} + \nu^{147}.$$

In all the cases, φ turns out to be locally conformal parallel with Lee 1-form $\tau = m\,\nu^7$. Now, for every Lie algebra the new basis of 1-forms:

$$e^1 = \nu^3, \quad e^2 = \nu^2, \quad e^3 = \nu^1, \quad e^4 = \nu^4, \quad e^5 = -\nu^6, \quad e^6 = \nu^5, \quad e^7 = \nu^7,$$

expresses φ in our canonical way (1), and the structure equations of $\mathfrak{cp}_s^m$, $1 \le s \le 7$, result as above. □

A quick inspection of the Lie algebras $\mathfrak{cp}_s^m$ listed in Proposition 2.1 reveals that each of them is determined by two tuples: one containing the eigenvalues $(\eta_1, \dots, \eta_6)$ of the derivation D and other one including the non-identically zero structure constants $\big(c^1_{36}, c^1_{45}, c^2_{35}, c^2_{46}, c^4_{26}, c^5_{23}\big)$ of the underlying 6-dimensional Lie algebra. In Table 2 we set both tuples for each case.

Now, we shall study solutions to the Laplacian flow on every Lie algebra $\mathfrak{cp}_s^m$. As we mentioned before, we assume a family of G_2-structures $\varphi(t)$ given by (6) where the unknown data are some differentiable real functions $f_i(t)$ depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is a real open interval. Observe that in fact, the functions $f_i(t)$ are positive. The basis $x^i(t) = f_i(t)e^i$ is adapted to the G_2-structure at any t, and the structure equations for any of the Lie algebras $\mathfrak{cp}_s^m$ depend on the functions and defining parameters of the algebras contained in Table 2:

$$\begin{cases} dx^1 = \dfrac{\eta_1}{f_7(t)}\,x^{17} + c^1_{36}\dfrac{f_1(t)}{f_{36}(t)}\,x^{36} + c^1_{45}\dfrac{f_1(t)}{f_{45}(t)}\,x^{45}, & dx^5 = \dfrac{\eta_5}{f_7(t)}\,x^{57} + c^5_{23}\dfrac{f_5(t)}{f_{23}(t)}\,x^{23}, \\ dx^2 = \dfrac{\eta_2}{f_7(t)}\,x^{27} + c^2_{35}\dfrac{f_2(t)}{f_{35}(t)}\,x^{35} + c^2_{46}\dfrac{f_2(t)}{f_{46}(t)}\,x^{46}, & dx^6 = \dfrac{\eta_6}{f_7(t)}\,x^{67}, \\ dx^3 = \dfrac{\eta_3}{f_7(t)}\,x^{37}, & dx^7 = 0. \\ dx^4 = \dfrac{\eta_4}{f_7(t)}\,x^{47} + c^4_{26}\dfrac{f_4(t)}{f_{26}(t)}\,x^{26}. & \end{cases} \tag{11}$$

Table 2. Defining parameters of the Lie algebras $\mathfrak{cp}_s^m$

	$(\eta_1, \dots, \eta_6)$	$\left(c_{36}^1, c_{45}^1, c_{35}^2, c_{46}^2, c_{26}^4, c_{23}^5\right)$
$\mathfrak{cp}_1^m$	$(-m, -m, -m, -m, -m, -m)$	$(0, 0, 0, 0, 0, 0)$
$\mathfrak{cp}_2^m$	$\left(-\frac{4m}{3}, -m, -\frac{2m}{3}, -m, -m, -\frac{2m}{3}\right)$	$\left(\frac{2m}{3}, 0, 0, 0, 0, 0\right)$
$\mathfrak{cp}_3^m$	$\left(-\frac{3m}{2}, -m, -\frac{3m}{4}, -\frac{3m}{4}, -\frac{3m}{4}, -\frac{3m}{4}\right)$	$\left(\frac{m}{2}, \frac{m}{2}, 0, 0, 0, 0\right)$
$\mathfrak{cp}_4^m$	$\left(-\frac{7m}{5}, -\frac{6m}{5}, -\frac{4m}{5}, -\frac{3m}{5}, -\frac{4m}{5}, -\frac{3m}{5}\right)$	$\left(\frac{2m}{5}, \frac{2m}{5}, 0, -\frac{2m}{5}, 0, 0\right)$
$\mathfrak{cp}_5^m$	$\left(-\frac{5m}{4}, -\frac{5m}{4}, -m, -\frac{m}{2}, -\frac{3m}{4}, -\frac{3m}{4}\right)$	$\left(0, \frac{m}{2}, 0, -\frac{m}{2}, 0, 0\right)$
$\mathfrak{cp}_6^m$	$\left(-\frac{4m}{3}, -\frac{4m}{3}, -\frac{2m}{3}, -\frac{2m}{3}, -\frac{2m}{3}, -\frac{2m}{3}\right)$	$\left(\frac{m}{3}, \frac{m}{3}, \frac{m}{3}, -\frac{m}{3}, 0, 0\right)$
$\mathfrak{cp}_7^m$	$\left(-\frac{6m}{5}, -\frac{3m}{5}, -\frac{3m}{5}, -\frac{6m}{5}, -\frac{6m}{5}, -\frac{3m}{5}\right)$	$\left(\frac{2m}{5}, 0, 0, 0, \frac{2m}{5}, \frac{2m}{5}\right)$

Since we want to solve the LCP-flow (9), we need to solve two equations. Let us start looking for necessary and sufficient conditions on the evolution functions $f_i(t)$ in order to preserve the locally conformal parallel condition, i.e, $\varphi(t) \in \mathcal{X}_4$, that we state in a more restrictive version imposing that the Lee 1-form remains constant along the flow:

Proposition 2.2. *The family of* G_2*-structures* $\varphi(t)$ *given by* (6) *satisfies*

$$d\varphi(t) = 3m\, e^7 \wedge \varphi(t), \quad d *_t \varphi(t) = 4m\, e^7 \wedge *_t\varphi(t), \tag{12}$$

and in particular remains locally conformal parallel if and only if the evolution functions $f_i(t)$, $1 \le i \le 7$, *satisfy the following conditions:*

- $\mathfrak{cp}_1^m$*: For any* $f_i(t)$, $i = 1, \dots, 7$.
- $\mathfrak{cp}_2^m$*:* $f_{17}(t) = f_{36}(t)$.
- $\mathfrak{cp}_3^m$*:* $f_{17}(t) = f_{36}(t) = f_{45}(t)$.
- $\mathfrak{cp}_4^m$*:* $f_{17}(t) = f_{36}(t) = f_{45}(t),\ f_{27}(t) = f_{46}(t)$.
- $\mathfrak{cp}_5^m$*:* $f_{17}(t) = f_{45}(t),\ f_{27}(t) = f_{46}(t)$.
- $\mathfrak{cp}_6^m$*:* $f_{17}(t) = f_{36}(t) = f_{45}(t),\ f_{27}(t) = f_{35}(t) = f_{46}(t)$.
- $\mathfrak{cp}_7^m$*:* $f_{17}(t) = f_{36}(t),\ \ f_{23}(t) = f_{57}(t),\ \ f_{26}(t) = f_{47}(t)$.

Proof. Let us start computing $d\varphi(t)$ using the general structure equations (11). Directly:

$$\begin{aligned} d\varphi(t) = {} & x^{1357}\left[\frac{\eta_1+\eta_3+\eta_5}{f_7(t)} - c_{35}^2\frac{f_2(t)}{f_{35}(t)}\right] - x^{1467}\left[\frac{\eta_1+\eta_4+\eta_6}{f_7(t)} + c_{46}^2\frac{f_2(t)}{f_{46}(t)}\right] \\ & - x^{2367}\left[\frac{\eta_2+\eta_3+\eta_6}{f_7(t)} - c_{36}^1\frac{f_1(t)}{f_{36}(t)} - c_{26}^4\frac{f_4(t)}{f_{26}(t)} - c_{23}^5\frac{f_5(t)}{f_{23}(t)}\right] \\ & - x^{2457}\left[\frac{\eta_2+\eta_4+\eta_5}{f_7(t)} - c_{45}^1\frac{f_1(t)}{f_{45}(t)}\right]. \end{aligned}$$

Now, the equation $d\varphi(t) = 3m\, e^7 \wedge \varphi(t) = \frac{3m}{f_7(t)}\, x^7 \wedge \varphi(t)$ is equivalent to the following system of equations:

$$\begin{cases} (\eta_1 + \eta_3 + \eta_5 + 3m)\, f_{135}(t) = c_{35}^2\, f_{127}(t), \\ (\eta_1 + \eta_4 + \eta_6 + 3m)\, f_{146}(t) = -c_{46}^2\, f_{127}(t), \\ (\eta_2 + \eta_4 + \eta_5 + 3m)\, f_{245}(t) = c_{45}^1\, f_{127}(t), \\ (\eta_2 + \eta_3 + \eta_6 + 3m)\, f_{236}(t) = c_{36}^1\, f_{127}(t) + c_{26}^4\, f_{347}(t) + c_{23}^5\, f_{567}(t). \end{cases} \tag{13}$$

Similar computations for $d *_t \varphi(t)$ yield:

$$\begin{aligned} d *_t \varphi(t) = & -x^{12347}\left[\frac{\eta_1+\eta_2+\eta_3+\eta_4}{f_7(t)} - c_{23}^5\frac{f_5(t)}{f_{23}(t)}\right] \\ & -x^{12567}\left[\frac{\eta_1+\eta_2+\eta_5+\eta_6}{f_7(t)} - c_{26}^4\frac{f_4(t)}{f_{26}(t)}\right] \\ & -x^{34567}\left[\frac{\eta_3+\eta_4+\eta_5-\eta_6}{f_7(t)} - c_{36}^1\frac{f_1(t)}{f_{36}(t)}\right. \\ & \left. -c_{45}^1\frac{f_1(t)}{f_{45}(t)} - c_{35}^2\frac{f_2(t)}{f_{35}(t)} + c_{46}^2\frac{f_2(t)}{f_{46}(t)}\right]. \end{aligned}$$

Again, solving the equation $d *_t \varphi(t) = 4m\, e^7 \wedge *_t\varphi(t) = \frac{4m}{f_7(t)} x^7 \wedge *_t\varphi(t)$ is equivalent to solve the system of equations:

$$\begin{cases} \eta_1+\eta_2+\eta_3+\eta_4+4m = c_{23}^5\dfrac{f_{57}(t)}{f_{23}(t)}, \\ \eta_1+\eta_2+\eta_5+\eta_6+4m = c_{26}^4\dfrac{f_{47}(t)}{f_{26}(t)}, \\ \eta_3+\eta_4+\eta_5+\eta_6+4m = c_{36}^1\dfrac{f_{17}(t)}{f_{36}(t)} + c_{45}^1\dfrac{f_{17}(t)}{f_{45}(t)} + c_{35}^2\dfrac{f_{27}(t)}{f_{35}(t)} - c_{46}^2\dfrac{f_{27}(t)}{f_{46}(t)}. \end{cases} \tag{14}$$

The final result is obtained by substituting the defining parameters of the Lie algebras $\mathfrak{cp}_s^m$ listed in Table 2 in both the expressions (13) and (14). □

After solving $\varphi(t) \in \mathcal{X}_4$, let us focus on the evolution equation $\dfrac{d\varphi(t)}{dt} = \Delta_t\varphi(t)$. Next, we get a generic expression of the Laplacian $\Delta_t\varphi(t)$ suitable for any of the Lie algebras $\mathfrak{cp}_s^m$.

Proposition 2.3. *Let $\varphi(t)$ be a family of* G_2*-structures given by* (6) *and remaining locally conformal parallel in the sense of* (12)*, for each Lie algebra $\mathfrak{cp}_s^m$ the Laplacian $\Delta_t\varphi(t)$ is given by:*

$$\Delta_t\varphi(t) = \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)\Delta_{ijk}\, x^{ijk}$$

where

$$\Delta_{127} = \frac{m}{f_7^2}\left[3(4m+\eta_3+\eta_4+\eta_5+\eta_6) + 4(\eta_1+\eta_2)\right],$$

$$\Delta_{347} = \frac{m}{f_7^2}\left[\frac{6m}{5}\,\delta_7 + 4\,(\eta_3+\eta_4)\right],$$
$$\Delta_{567} = \frac{m}{f_7^2}\left[\frac{6m}{5}\,\delta_7 + 4\,(\eta_5+\eta_6)\right],$$
$$\Delta_{135} = \frac{m}{f_7^2}\left[\frac{4m}{3}\,\delta_6 + 3\,(\eta_2+\eta_4+\eta_6)\right],$$
$$\Delta_{146} = \frac{m}{f_7^2}\left[\frac{8m}{5}\,\delta_4 + 2m\,\delta_5 + \frac{4m}{3}\,\delta_6 + 3\,(\eta_2+\eta_3+\eta_5)\right],$$
$$\Delta_{236} = \frac{m}{f_7^2}\left[\frac{8m}{3}\,\delta_2 + 2m\,\delta_3 + \frac{8m}{5}\,\delta_4 + \frac{4m}{3}\,\delta_6 + \frac{24m}{5}\,\delta_7 + 3\,(\eta_1+\eta_4+\eta_5)\right],$$
$$\Delta_{245} = \frac{m}{f_7^2}\left[2m\,\delta_3 + \frac{8m}{5}\,\delta_4 + 2m\,\delta_5 + \frac{4m}{3}\,\delta_6 - 3\,(\eta_1+\eta_3+\eta_6)\right], \qquad (15)$$

and $\delta_s = \begin{cases} 1, & \textit{if } \mathfrak{g} \cong \mathfrak{cp}_s^m, \\ 0, & \textit{if } \mathfrak{g} \ncong \mathfrak{cp}_s^m. \end{cases}$

Proof. The Laplacian operator on 3-forms is defined as: $\Delta_t\varphi(t) = -d*d*\varphi(t) + *d*d\varphi(t)$. Taking into account the conformally parallel conditions (12) expressed in terms of the orthonormal basis $\{x_i\}_{i=1}^7$, the Laplacian of $\varphi(t)$ can be computed as:

$$\begin{aligned}\Delta_t\varphi(t) &= \frac{m}{f_7^2(t)}\left[-4\left(d*\left(x^7\wedge *\varphi(t)\right)\right) + 3\left(*d*\left(x^7\wedge\varphi(t)\right)\right)\right] \\ &= \frac{m}{f_7^2(t)}\left[-4\left(d*\left(x^{12347}+x^{12567}+x^{34567}\right)\right)\right. \\ &\quad \left.+3\left(*d*\left(-x^{1357}+x^{1467}+x^{2367}+x^{2457}\right)\right)\right] \\ &= \frac{m}{f_7^2(t)}\left[-4\left(d\left(x^{12}+x^{34}+x^{56}\right)\right) + 3\left(*d\left(x^{136}+x^{145}+x^{235}-x^{246}\right)\right)\right].\end{aligned}$$

If we apply (11) and the Hodge star operator in the second summand, we obtain the following expression

$$\begin{aligned}\Delta_t\varphi(t) = \frac{m}{f_7^2(t)}&\left[3\left(c_{36}^1\frac{f_1(t)}{f_{36}(t)} + c_{45}^1\frac{f_1(t)}{f_{45}(t)} + c_{35}^2\frac{f_2(t)}{f_{35}(t)} - c_{46}^2\frac{f_2(t)}{f_{46}(t)}\right)\right. \\ &\left.+4\left(\frac{\eta_1+\eta_2}{f_7(t)}\right)\right]x^{127} \\ &+\frac{m}{f_7^2(t)}\left[3\,c_{26}^4\frac{f_4(t)}{f_{26}(t)} + 4\left(\frac{\eta_3+\eta_4}{f_7(t)}\right)\right]x^{347} \\ &+\frac{m}{f_7^2(t)}\left[3\,c_{23}^5\frac{f_5(t)}{f_{23}(t)} + 4\left(\frac{\eta_5+\eta_6}{f_7(t)}\right)\right]x^{567} \\ &+\frac{m}{f_7^2(t)}\left[3\left(\frac{\eta_2+\eta_4+\eta_6}{f_7(t)}\right) + 4\,c_{35}^2\frac{f_2(t)}{f_{35}(t)}\right]x^{135}\end{aligned}$$

$$+\frac{m}{f_7^2(t)}\left[-3\left(\frac{\eta_2+\eta_3+\eta_5}{f_7(t)}\right)+4\,c_{46}^2\frac{f_2(t)}{f_{46}(t)}\right]x^{146}$$
$$+\frac{m}{f_7^2(t)}\left[-3\left(\frac{\eta_1+\eta_4+\eta_5}{f_7(t)}\right)\right.$$
$$\left.-4\left(c_{36}^1\frac{f_1(t)}{f_{36}(t)}+c_{26}^4\frac{f_4(t)}{f_{26}(t)}+c_{23}^5\frac{f_5(t)}{f_{23}(t)}\right)\right]x^{236}$$
$$+\frac{m}{f_7^2(t)}\left[-3\left(\frac{\eta_1+\eta_3+\eta_6}{f_7(t)}\right)-4\,c_{45}^1\frac{f_1(t)}{f_{45}(t)}\right]x^{245}.$$

To get the final expression just apply Proposition 2.2 together with the defining parameters of the Lie algebras collected in Table 2. □

Remark 2.4. We notice that for any family of G_2-structures $\varphi(t)$ given by (6) and any $(i, j, k) \in A \cup B$ the expressions of $f_7^2(t)\Delta_{ijk}$ obtained in (15) depend only on the defining parameters of the Lie algebra $\mathfrak{cp}_s^m$ and not on the functions $f_i(t)$.

3. Long time solutions of the Laplacian flow of an LCP G_2-structure

In this section we obtain long time solutions for the Laplacian flow on the solvable Lie groups S_s, $s = 1, \dots, 7$, where S_s has underlying Lie algebra $\mathfrak{cp}_s^m$ described in Proposition 2.1 in terms of a basis $\{e^1, \dots, e^7\}$ such that the canonical 3-form φ_0 given by (1) is an LCP G_2-structure. We divide our study starting by the solvable Lie group S_1 as the results obtained on it guide the method on the rest of cases.

Theorem 3.1. *Let S_1 be a solvable Lie group with underlying Lie algebra $\mathfrak{cp}_1^m$. The family of* G_2*-structures given by:*

$$\varphi(t) = \left(1-4m^2t\right)^2\left(e^{12}+e^{34}+e^{56}\right)\wedge e^7$$
$$+\left(1-4m^2t\right)^{\frac{9}{4}}\left(e^{135}-e^{146}-e^{236}-e^{245}\right),\quad t\in I=\left(-\infty,\frac{1}{4m^2}\right)$$

is the unique solution for the Laplacian LCP-flow (9). Moreover, the underlying metric $g(t)$ is Einstein at any $t \in I$ and converges to a flat metric as t goes to $-\infty$.

Proof. Taking into account (11) and the defining parameters of the Lie algebra $\mathfrak{cp}_1^m$ given in Table 2, the Maurer–Cartan equations in the adapted basis $\{x^1, \dots, x^7\}$ are:

$$\begin{cases} dx^k = -\frac{m}{f_7}\,x^k\wedge x^7, \quad 1\le k\le 6,\\ dx^7 = 0.\end{cases}$$

Proposition 2.2 shows that for $\mathfrak{cp}_1^m$ the family of G_2-structures $\varphi(t)$ given by (6) remains locally conformal parallel regardless of the evolution functions $f_i(t)$. Then, we only need to solve the evolution equation $\dfrac{d\varphi(t)}{dt} = \Delta_t\varphi(t)$.

We get the expression of the Laplacian $\Delta_t\varphi(t)$ substituting the defining parameters of the Lie algebra $\mathfrak{cp}_1^m$ provided in Table 2 in the generic formula given in Proposition 2.3:

$$\Delta_t\varphi(t) = \frac{-m^2}{f_7^2(t)}\left[8\left(x^{127}+x^{347}+x^{567}\right)+9\left(x^{135}-x^{146}-x^{236}-x^{245}\right)\right].$$

Now, the equalities:

$$\Delta_{127}=\Delta_{347}=\Delta_{567}=\frac{-8m^2}{f_7^2(t)},\quad \Delta_{135}=\Delta_{146}=\Delta_{236}=\Delta_{245}=\frac{-9m^2}{f_7^2(t)},$$

imply respectively by Lemma 1.3 part (i) that $f_{12}=f_{34}=f_{56}$ and $f_{135}=f_{146}=f_{236}=f_{245}$. From the first group we get $f_4(t)=\frac{f_{12}(t)}{f_3(t)}$ and $f_6(t)=\frac{f_{12}(t)}{f_5(t)}$, thus, substituting in the second one we get $f_1^2(t)=f_2^2(t)=f_3^2(t)=f_5^2(t)$. Furthermore, as $f_i(t)>0$, we conclude that $f_i(t)=f(t)$ for any $1\le i\le 6$.

At this point, solving the evolution equation (8) reduces to solve the following system of two differential equations with unknowns $f(t)$ and $f_7(t)$:

$$\begin{cases} \dfrac{-8m^2}{f_7^2(t)}=\Delta_{127}=\dfrac{f'_{127}}{f_{127}}=\dfrac{d}{dt}\ln(f_{127})=\dfrac{d}{dt}[\ln(f_1(t))+\ln(f_2(t))+\ln(f_7(t))]\\ \quad =2\dfrac{f'(t)}{f(t)}+\dfrac{f_7'(t)}{f_7(t)},\\ \dfrac{-9m^2}{f_7^2(t)}=\Delta_{135}=\dfrac{f'_{135}}{f_{135}}=\dfrac{d}{dt}\ln(f_{135})=\dfrac{d}{dt}[\ln(f_1(t))+\ln(f_3(t))+\ln(f_5(t))]\\ \quad =3\dfrac{f'(t)}{f(t)}, \end{cases}$$

which is equivalent to:

$$\begin{cases} \dfrac{-2m^2}{f_7^2(t)}=\dfrac{f_7'(t)}{f_7(t)},\\ \dfrac{-3m^2}{f_7^2(t)}=\dfrac{f'(t)}{f(t)}. \end{cases}$$

The first equation involves only $f_7(t)$ and can be explicitly solved:

$$f_7(t)f_7'(t)=-2m^2 \Longrightarrow f_7(t)=\left(-4m^2t+C\right)^{1/2}.$$

Moreover, using the fact that $f_7(0)=1$, we get that $C=1$ and $f_7(t)=\left(1-4m^2t\right)^{1/2}$. With this value for $f_7(t)$, it is also possible to solve explicitly the second equation:

$$\frac{-3m^2}{1-4m^2t}=\frac{f'(t)}{f(t)}\Longrightarrow \frac{3}{4}\ln\left(1-4m^2t\right)=\ln f(t)+C.$$

Again, the value of C is determined imposing the initial condition $f(0) = 1$, obtaining that $f(t) = \left(1 - 4m^2t\right)^{3/4}$. The domains of the functions $f(t)$ and $f_7(t)$ imply that the family $\varphi(t)$ of G_2-structures is defined for any $t \in I = \left(-\infty, \frac{1}{4m^2}\right)$.

Concerning the metric, it turns out that the non-vanishing components of the curvature tensor $R_{ijkl} = g(R(x_i, x_j)x_k, x_l)$ at any $t \in I$ are (modulo its symmetry properties):

$$R_{ijji} = -\frac{m^2}{1 - 4m^2t} \quad \text{for any } 1 \le i < j \le 7.$$

Thus, $\lim_{t\to-\infty} R(g_t) = 0$. Moreover, an standard computation shows that the Ricci tensor $Ric(g_t)_{ij} = \sum_{k=1}^{7} R_{kijk}$ satisfies

$$Ric(g_t) = -\frac{6m^2}{1 - 4m^2t}\, g_t,$$

that is, g_t is Einstein concluding the proof. □

For the rest of the Lie algebras $\mathfrak{cp}_s^m$, we obtain the following explicit solutions:

Theorem 3.2. *Let S_s be a solvable Lie group with underlying Lie algebra $\mathfrak{cp}_s^m$. The family of* G_2*-structures given below is a solution for the Laplacian flow:*

- $\mathfrak{cp}_2^m$*: For $t \in \left(-\infty, \frac{3}{10m^2}\right)$,*

$$\varphi(t) = \left(1 - \frac{10}{3}m^2t\right)^{\frac{11}{5}} \left(e^{127} - e^{236}\right) + \left(1 - \frac{10}{3}m^2t\right)^{2} \left(e^{347} - e^{567}\right) + \left(1 - \frac{10}{3}m^2t\right)^{\frac{12}{5}} \left(e^{135} - e^{146} - e^{245}\right).$$

- $\mathfrak{cp}_3^m$*: For $t \in \left(-\infty, \frac{1}{3m^2}\right)$,*

$$\varphi(t) = \left(1 - 3m^2t\right)^{\frac{7}{3}} \left(e^{127} - e^{236} - e^{245}\right) + \left(1 - 3m^2t\right)^{2} \left(e^{347} + e^{567}\right) + \left(1 - 3m^2t\right)^{\frac{5}{2}} \left(e^{135} - e^{146}\right).$$

- $\mathfrak{cp}_4^m$*: For $t \in \left(-\infty, \frac{5}{14m^2}\right)$,*

$$\varphi(t) = \left(1 - \frac{14}{5}m^2t\right)^{\frac{17}{7}} \left(e^{127} - e^{146} - e^{236} - e^{245}\right) + \left(1 - \frac{14}{5}m^2t\right)^{2} \left(e^{347} + e^{567}\right) + \left(1 - \frac{14}{5}m^2t\right)^{\frac{18}{7}} e^{135}.$$

- $\mathfrak{cp}_5^m$: *For* $t \in \left(-\infty, \frac{1}{3m^2}\right)$,

$$\varphi(t) = \left(1 - 3m^2 t\right)^{\frac{4}{3}} \left(e^{127} - e^{146} - e^{245}\right) + \left(1 - 3m^2 t\right)^2 \left(e^{347} + e^{567}\right) + \left(1 - 3m^2 t\right)^{\frac{5}{2}} \left(e^{135} - e^{236}\right).$$

- $\mathfrak{cp}_6^m$: *For* $t \in \left(-\infty, \frac{3}{8m^2}\right)$,

$$\varphi(t) = \left(1 - \frac{8}{3}m^2 t\right)^{\frac{5}{2}} \left(e^{127} + e^{135} - e^{146} - e^{236} - e^{245}\right) + \left(1 - \frac{8}{3}m^2 t\right)^2 \left(e^{347} + e^{567}\right).$$

- $\mathfrak{cp}_7^m$: *For* $t \in \left(-\infty, \frac{5}{14m^2}\right)$,

$$\varphi(t) = \left(1 - \frac{14}{5}m^2 t\right)^{\frac{15}{7}} \left(e^{127} + e^{347} + e^{567} - e^{236}\right) + \left(1 - \frac{14}{5}m^2 t\right)^{\frac{18}{7}} \left(e^{135} - e^{146} - e^{245}\right).$$

Proof. Inspired by the solution to the Laplacian flow on the solvable Lie group S_1 obtained in Theorem 3.1 we will consider families of G_2-structures of type (6) on the rest of Lie groups S_s where the evolution functions $f_i^s(t)$ are specifically given by:

$$f_i^s(t) = \left(1 - \alpha_s\, m^2\, t\right)^{\beta_i^s}, \quad \text{for any} \quad 1 \le i \le 7, \tag{16}$$

hence, for each Lie algebra $\mathfrak{cp}_s^m$, the unknowns are now $\alpha_s \in \mathbb{R}^*$ and $\beta_i^s \in \mathbb{R}$, $i = 1, \ldots, 7$. Now, Proposition 2.2 states necessary and sufficient conditions for the property being LCP to be preserved during the flow, which under the assumption (16) transform into relations involving the β_i coefficients as follows:

$$\begin{aligned}
&\mathfrak{cp}_2^m : \beta_1 + \beta_7 = \beta_3 + \beta_6;\\
&\mathfrak{cp}_3^m : \beta_1 + \beta_7 = \beta_3 + \beta_6 = \beta_4 + \beta_5;\\
&\mathfrak{cp}_4^m : \beta_1 + \beta_7 = \beta_3 + \beta_6 = \beta_4 + \beta_5, \quad \beta_2 + \beta_7 = \beta_4 + \beta_6;\\
&\mathfrak{cp}_5^m : \beta_1 + \beta_7 = \beta_4 + \beta_5, \quad \beta_2 + \beta_7 = \beta_4 + \beta_6;\\
&\mathfrak{cp}_6^m : \beta_1 + \beta_7 = \beta_3 + \beta_6 = \beta_4 + \beta_5, \quad \beta_2 + \beta_7 = \beta_3 + \beta_5 = \beta_4 + \beta_6;\\
&\mathfrak{cp}_7^m : \beta_1 + \beta_7 = \beta_3 + \beta_6, \quad \beta_2 + \beta_3 = \beta_5 + \beta_7, \quad \beta_2 + \beta_6 = \beta_4 + \beta_7.
\end{aligned} \tag{17}$$

In addition, system (8) reduces to

$$\Delta_{ijk} = \frac{(f_{ijk})'}{f_{ijk}} = \frac{-\alpha m^2(\beta_i + \beta_j + \beta_k)}{(1 - \alpha m^2 t)},$$

where the unknowns are α and $\beta_1, \ldots, \beta_7$.

Explicitly, taking the Δ_{ijk} coefficients given in Proposition 2.3:

$$\beta_1 + \beta_2 + \beta_7 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} \left[3(4m + \eta_3 + \eta_4 + \eta_5 + \eta_6) + 4(\eta_1 + \eta_2)\right],$$

$$\beta_3 + \beta_4 + \beta_7 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} \left[\frac{6m}{5}\,\delta_7 + 4\,(\eta_3 + \eta_4)\right],$$

$$\beta_5 + \beta_6 + \beta_7 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} \left[\frac{6m}{5}\,\delta_7 + 4\,(\eta_5 + \eta_6)\right],$$

$$\beta_1 + \beta_3 + \beta_5 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} \left[\frac{4m}{3}\,\delta_6 + 3\,(\eta_2 + \eta_4 + \eta_6)\right],$$

$$\beta_1 + \beta_4 + \beta_6 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} \left[\frac{8m}{5}\,\delta_4 + 2m\,\delta_5 + \frac{4m}{3}\,\delta_6 + 3\,(\eta_2 + \eta_3 + \eta_5)\right],$$

$$\begin{aligned}\beta_2 + \beta_3 + \beta_6 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} &\left[\frac{8m}{3}\,\delta_2 + 2m\,\delta_3 + \frac{8m}{5}\,\delta_4 + \frac{4m}{3}\,\delta_6\right. \\ &\left. + \frac{24m}{5}\,\delta_7 + 3\,(\eta_1 + \eta_4 + \eta_5)\right],\end{aligned}$$

$$\begin{aligned}\beta_2 + \beta_4 + \beta_5 = -\frac{\left(1 - \alpha m^2 t\right)^{1-2\beta_7}}{\alpha m} &\left[2m\,\delta_3 + \frac{8m}{5}\,\delta_4 + 2m\,\delta_5\right. \\ &\left. + \frac{4m}{3}\,\delta_6 - 3\,(\eta_1 + \eta_3 + \eta_6)\right].\end{aligned}$$

Clearly, the latter system admits solution only if $\beta_7 = \frac{1}{2}$. Now, for each Lie algebra $\mathfrak{cp}_s^m$ the values of α and $\beta_1, \ldots, \beta_6$ result of solving the system that yields substituting above the concrete values of $\eta_1, \ldots, \eta_6$ listed in Table 2 joint with the corresponding relations (17) involving the preservation of the LCP condition during the flow. The values of the solution parameters are listed in Table 3 and the resulting solutions $\varphi(t)$ are picked in the statement of the theorem.

□

Table 3. Defining parameters of the functions $f_i(t) = (1 - \alpha m^2 t)^{\beta_i}$

Lie group	α	$(\beta_1, \ldots, \beta_7)$	Lie group	α	$(\beta_1, \ldots, \beta_7)$
S_2	$\frac{10}{3}$	$\left(\frac{9}{10}, \frac{4}{5}, \frac{7}{10}, \frac{4}{5}, \frac{4}{5}, \frac{7}{10}, \frac{1}{2}\right)$	S_5	3	$\left(\frac{11}{12}, \frac{11}{12}, \frac{5}{6}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$
S_3	3	$\left(1, \frac{5}{6}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$	S_6	$\frac{8}{3}$	$\left(1, 1, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$
S_4	$\frac{14}{5}$	$\left(1, \frac{13}{14}, \frac{11}{14}, \frac{5}{7}, \frac{11}{14}, \frac{5}{7}, \frac{1}{2}\right)$	S_7	$\frac{14}{5}$	$\left(\frac{13}{14}, \frac{10}{14}, \frac{10}{14}, \frac{13}{14}, \frac{13}{14}, \frac{10}{14}, \frac{1}{2}\right)$

Remark 3.3. The non-vanishing elements (up to symmetries) of the Riemannian curvature of g_t are tabulated in Tables 4 and 5 in "Appendix", where the coefficients $C_1,\ C_2,\ C_3,\ C_4$ are given in terms of the parameter α by:

$$C_1 = \frac{-6m^2}{1 - \alpha\, m^2 t}, \quad C_2 = \frac{-m^2}{3}\left(\frac{1}{1 - \alpha\, m^2 t}\right),$$
$$C_3 = \frac{-m^2}{4}\left(\frac{1}{1 - \alpha\, m^2 t}\right), \quad C_4 = \frac{-m^2}{5}\left(\frac{1}{1 - \alpha\, m^2 t}\right).$$

A direct computation of the Ricci tensors in the orthonormal basis $\{x_i(t)\}_{i=1}^7$ shows that:

$$\begin{aligned}
\mathfrak{cp}_2^m &: \mathrm{Ric}(g_t) = C_2\, diag(22, 17, 12, 17, 17, 12, 17),\\
\mathfrak{cp}_3^m &: \mathrm{Ric}(g_t) = C_3\, diag(32, 22, 17, 17, 17, 17, 22),\\
\mathfrak{cp}_4^m &: \mathrm{Ric}(g_t) = C_4\, diag(37, 32, 22, 17, 22, 17, 27),\\
\mathfrak{cp}_5^m &: \mathrm{Ric}(g_t) = C_3\, diag(27, 27, 22, 12, 17, 17, 22),\\
\mathfrak{cp}_6^m &: \mathrm{Ric}(g_t) = C_2\, diag(21, 21, 11, 11, 11, 11, 16),\\
\mathfrak{cp}_7^m &: \mathrm{Ric}(g_t) = C_4\, diag(32, 17, 17, 32, 32, 17, 27).
\end{aligned}$$

Hence, unlike the case of S_1, none of the metrics listed in Theorem 3.2 obtained as solutions of the Laplacian LCP-flow for S_s, $s = 2, \ldots, 7$, are Einstein.

Proposition 3.4. *The* G_2*-structures obtained in Theorems* 3.1 *and* 3.2 *are Laplacian solitons of shrinking type.*

Proof. Recall that a G_2-structure φ is called *Laplacian soliton* if it satisfies the equation

$$\Delta\varphi = \lambda\varphi + \mathcal{L}_X\varphi,$$

for some real number λ and some vector field X. Depending on the sign of λ, Laplacian solitons are called *shrinking* (if $\lambda < 0$); *steady* (if $\lambda = 0$) or *expanding* (if $\lambda > 0$).

In the left-invariant setting, the Lie derivative of a 3-form Ω can be computed following the formula:

$$\mathcal{L}_X\Omega(Y_1, Y_2, Y_3) = -\Omega([X, Y_1], Y_2, Y_3) - \Omega(Y_1, [X, Y_2], Y_3) - \Omega(Y_1, Y_2, [X, Y_3]),$$

where $Y_1,\ Y_2,\ Y_3$ are invariant vector fields.

In our case, consider the left-invariant vector field $X = -\frac{m}{f_7(t)} X_7$, where X_7 denotes the dual of the 1-form x^7. Then, taking into account the generic structure equations (11) and the formula above we get an expression of the Lie derivative $\mathcal{L}_X\varphi(t)$ in terms of the defining parameters $\eta_1, \ldots, \eta_6$ of the Lie algebras:

$$\mathcal{L}_X\varphi(t) = \frac{m}{f_7^2(t)} \Big[(\eta_1+\eta_2)\, x^{127} + (\eta_3+\eta_4)\, x^{347} + (\eta_5+\eta_6)\, x^{567} + (\eta_1+\eta_3+\eta_5)\, x^{135} \\ -(\eta_1+\eta_4+\eta_6)\, x^{146} - (\eta_2+\eta_3+\eta_6)\, x^{236} - (\eta_2+\eta_4+\eta_5)\, x^{245} \Big]. \qquad (18)$$

Now, if we compute $\Delta\varphi(t) - \mathcal{L}_X\varphi(t)$, using (18), Proposition 2.3 and Table 2, we obtain:

$$\Delta\varphi(t) - \mathcal{L}_X\varphi(t) = -\frac{m^2}{f_7^2(t)}\varphi(t) \left(6\,\delta_1 + 5\,\delta_2 + \frac{9}{2}\,\delta_3 + \frac{21}{5}\,\delta_4 + \frac{9}{2}\,\delta_5 + 4\,\delta_6 + \frac{21}{5}\,\delta_7 \right),$$

so they are Laplacian solitons. Moreover, since the constant λ_i is negative in all cases, the solitons are of shrinking type. □

4. Long time solutions of the Laplacian coflow of an LCP G_2-structure

In this section we consider the Laplacian coflow (10). More concretely, we seek explicit solutions to the coflow on the set of solvable Lie groups endowed with a left-invariant LCP structure listed in Proposition 2.1.

We look for solutions within the families of invariant G_2-structures $\varphi(t)$ of type (6) depending on some unknown functions $f_i(t)$, $1 \leq i \leq 7$, in the same terms as it has been set in the paper. Then, at any $t \in I$, the 4-form $\psi(t) = *_t\varphi(t)$ involved in the evolution equation of the coflow can be expressed in terms of the adapted basis $\{x^i(t)\}_{i=1}^7$ as:

$$\begin{aligned}\psi(t) &= x^{3456} + x^{1256} + x^{1234} - x^{2467} + x^{2357} + x^{1457} + x^{1367} \\ &= f_{3456}(t)e^{3456} + f_{1256}(t)e^{1256} + f_{1234}(t)e^{1234} - f_{2467}(t)e^{2467} \\ &\quad + f_{2357}(t)e^{2357} + f_{1457}(t)e^{1457} + f_{1367}(t)e^{1367}.\end{aligned}$$

and the Laplacian of the 4-form $\psi(t)$:

$$\Delta_t\psi(t) = \sum_{(l,m,n,o)\in\mathcal{K}\cup\{(2,4,6,7)\}} \varepsilon(l,m,n,o)\Delta_{lmno}\, x^{lmno} + \sum_{(l,m,n,o)\notin\mathcal{K}\cup\{(2,4,6,7)\}} \Delta_{lmno} x^{lmno}. \qquad (19)$$

where $\mathcal{K} = \{(1,2,3,4), (1,2,5,6), (1,3,6,7), (1,4,5,7), (2,3,5,7), (3,4,5,6)\}$. The symbols $\varepsilon(l,m,n,o)$ are defined as:

$$\varepsilon(l,m,n,o) = \begin{cases} 1 & \text{if } (l,m,n,o) \in \mathcal{K}, \\ -1 & \text{if } (l,m,n,o) = (2,4,6,7); \end{cases}$$

Therefore, by a similar a similar argument as in the flow case, the first equation of the Laplacian LCP-coflow (10) becomes the system of differential equations:

$$\begin{cases} \Delta_{lmno} = \frac{-(f_{lmno})'}{f_{lmno}}, & \text{if } (l,m,n,o) \in \mathcal{K} \cup \{(2,4,6,7)\}, \\ \Delta_{lmno} = 0, & \text{otherwise.} \end{cases}$$

Concerning the preservation of the LCP property of $*_t\psi(t)$, functions $f_i(t)$ must satisfy the relations stated in Proposition 2.2. Now, in the next result we show that there is a correspondence between solutions of the ansatz type (6) of the Laplacian LCP-flow and the Laplacian LCP-coflow assuming that the functions are of potential type.

Theorem 4.1. *Let $\varphi(t)$ and $\widetilde{\varphi}(t)$ be two different families of* G_2*-structures on* $\mathfrak{cp}_s^m$ *with* $s = 1, \ldots, 7$*, given by* (6)*, where*

$$f_i^s(t) = \left(1 - \alpha\, m^2 t\right)^{\beta_i} \quad \textit{and} \quad \widetilde{f_i^s}(t) = \left(1 - \gamma\, m^2 t\right)^{\delta_i}, \quad \textit{for} \quad i = 1, \ldots, 7,$$

and $\beta_7 = \frac{1}{2}$ *and* $\delta_7 = \frac{1}{2}$*. If the defining parameters of the functions* $f_i^s(t)$ *and* $\widetilde{f_i^s}(t)$ *are related by:*

$$\gamma = \alpha \left(\frac{2 - \sum_{i=1}^{7} \beta_i}{2} \right), \quad \textit{and} \quad \delta_i = \frac{1}{2} + \frac{1 - 2\beta_i}{-2 + \sum_{j=1}^{7} \beta_j} \quad \textit{with} \quad i \in \{1, \ldots, 7\}, \tag{20}$$

then:

(i) $\varphi(t)$ *is LCP if and only if* $\widetilde{\varphi}(t)$ *is LCP.*
(ii) $\varphi(t)$ *solves the Laplacian LCP-flow* (9) *if and only if* $\widetilde{\psi}(t) = \widetilde{*}_t\widetilde{\varphi}(t)$ *solves the Laplacian LCP-coflow* (10).

Proof. We denote by $(l,m,n,o) \in \mathcal{K}$ the set of complementary indexes to $(i,j,k) \in A \cup B$ (see page 5), i.e. $(l,m,n,o) = \widehat{(i,j,k)} = (1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, \hat{k}, \ldots, 7)$. With this notation, (20) implies

$$\gamma\,(\delta_l + \delta_m + \delta_n + \delta_o) = -\alpha\,(\beta_i + \beta_j + \beta_k), \tag{21}$$

for all $(l,m,n,o) \in \mathcal{K}$ and $(l,m,n,o) = \widehat{(i,j,k)}$.

Now we prove the two statements of the theorem.

(i) Let us consider two pair of indexes $(i_1, j_1, k_1), (i_2, j_2, k_2) \in A \cup B$ such that they have a common index, let us say $k_1 = k_2 = k$. Under this hypothesis and making use of (21) the following identities hold:

$$\begin{aligned} \gamma(\delta_1 + \cdots + \delta_7) &= \gamma(\delta_1 + \cdots + \delta_7) \\ \gamma\left(\delta_{i_1} + \delta_{j_1} + \delta_k\right) + \gamma\left(\delta_{l_1} + \delta_{m_1} + \delta_{n_1} + \delta_{o_1}\right) &= \gamma\left(\delta_{i_2} + \delta_{j_2} + \delta_k\right) + \gamma\left(\delta_{l_2} + \delta_{m_2} + \delta_{n_2} + \delta_{o_2}\right) \\ \gamma\left(\delta_{i_1} + \delta_{j_1} + \delta_k\right) - \alpha\left(\beta_{i_1} + \beta_{j_1} + \beta_k\right) &= \gamma\left(\delta_{i_2} + \delta_{j_2} + \delta_k\right) - \alpha\left(\beta_{i_2} + \beta_{j_2} + \beta_k\right) \\ \gamma\left[\left(\delta_{i_2} + \delta_{j_2}\right) - \left(\delta_{i_1} + \delta_{j_1}\right)\right] &= -\alpha\left[\left(\beta_{i_1} + \beta_{j_1}\right) - \left(\beta_{i_2} + \beta_{j_2}\right)\right]. \end{aligned}$$

We observe that the relations (17) concerning the preservation of the LCP property are always of the form:

$$\beta_{i_1} + \beta_{j_1} = \beta_{i_2} + \beta_{j_2},$$

for a pair of indexes satisfying that $(i_1, j_1, k) \in A,\ (i_2, j_2, k) \in B$. The LCP conditions for $\widetilde{\varphi}(t)$ are exactly the same that the LCP conditions for $\varphi(t)$ interchanging the parameters β_i for δ_i. Therefore, considering the non-zero values of the parameter α of the solutions of the Laplacian flow (see Theorem 3.2) it is easy to see that $\gamma \neq 0$ in all the cases, hence we conclude that $\varphi(t)$ is LCP if and only if $\widetilde{\varphi}(t)$ is so.
(ii) Let $\varphi(t)$ and $\widetilde{\varphi}(t)$ be two families of G_2-structures (6) whose defining parameters of the functions $f_i^s(t)$ and $\widetilde{f_i^s}(t)$ are related by (20). Let $\varphi(t)$ be a solution of the Laplacian LCP-flow; we want to prove that $\widetilde{\psi}(t) = \widetilde{*}_t\widetilde{\varphi}(t)$ is a solution of the corresponding coflow. That $\tilde{\psi}(t)$ is solution of the coflow is equivalent to:

$$\left(\widetilde{f_7^s}(t)\right)^2 \tilde{\Delta}_{lmno} = \gamma m(\delta_l+\delta_m+\delta_n+\delta_o) \quad \text{for any} \quad (l, m, n, o) \in \mathcal{K} \cup \{(2, 4, 6, 7)\},$$

where we have used the same ideas as in the proof of Theorem 3.2, that is, the functions $\widetilde{f_i^s}(t)$ are of potential type and $\delta_7 = \frac{1}{2}$.

Firstly observe that, as the Hodge star operator commutes with the Laplacian operator, we have that $\Delta_t *_t \varphi(t) = *_t\Delta_t\varphi(t)$, hence, the coefficients of $\Delta_t\varphi(t)$ and $\Delta_t\psi(t)$ appearing in the linear combinations (7) and (19) are related by:

$$\Delta_{lmno} = \Delta_{ijk}, \quad \text{for any} \quad (i, j, k) \in A \cup B \text{ and } (l, m, n, o) = \widehat{(i, j, k)}.$$

This fact together with the non-dependence of the $\left(f_7^s(t)\right)^2 \Delta_{ijk}$ with respect to the specific chosen functions noticed in Remark 2.4 yields:

$$\left(\widetilde{f_7^s}(t)\right)^2 \tilde{\Delta}_{lmno} = \left(\widetilde{f_7^s}(t)\right)^2 \tilde{\Delta}_{ijk} = \left(f_7^s(t)\right)^2 \Delta_{ijk},$$
$$\text{for any} \quad (i, j, k) \in A \cup B \quad \text{and} \quad (l, m, n, o) = \widehat{(i, j, k)}.$$

Now, since $\varphi(t)$ is solution of the flow then we have:

$$\left(f_7^s(t)\right)^2 \Delta_{ijk} = -\alpha m(\beta_i + \beta_j + \beta_k) \quad \text{for any } (i, j, k) \in A \cup B.$$

Therefore, bearing in mind (21) the following sequence of identities hold:

$$\left(\widetilde{f_7^s}(t)\right)^2 \widetilde{\Delta}_{lmno} = \left(f_7^s(t)\right)^2 \Delta_{ijk} = -\alpha m(\beta_i + \beta_j + \beta_k) = \gamma m(\delta_l + \delta_m + \delta_n + \delta_o),$$

for every $(l, m, n, o) \in \mathcal{K} \cup \{(2, 4, 6, 7)\}$, that is, $\widetilde{\psi}(t)$ is a solution of the coflow.

The converse of the statement is basically the same and we omit the proof. □

As a consequence of the previous theorem, for every Lie group S_s we provide in the following corollary an explicit solution of the LCP-coflow based on the defining parameters of the solutions of the flow contained in Table 3.

Corollary 4.2. *Let S_s be a solvable Lie group with underlying Lie algebra $\mathfrak{cp}_s^m$. The family of* G_2*-structures given below is solution for the Laplacian coflow:*

- $\mathfrak{cp}_1^m$: *For* $t \in \left(-\frac{1}{6m^2}, \infty\right)$,

$$\varphi(t) = \left(1+6m^2t\right)^{\frac{7}{6}} \left(e^{127}+e^{347}+e^{567}\right) + \left(1+6m^2t\right)\left(e^{135}-e^{146}-e^{236}-e^{245}\right).$$

- $\mathfrak{cp}_2^m$: *For* $t \in \left(-\frac{3}{16m^2}, \infty\right)$,

$$\varphi(t) = \left(1+\frac{16}{3}m^2t\right)^{\frac{17}{16}} \left(e^{127}-e^{236}\right) + \left(1+\frac{16}{3}m^2t\right)^{\frac{19}{16}} \left(e^{347}+e^{567}\right) + \left(1+\frac{16}{3}m^2t\right)^{\frac{15}{16}} \left(e^{135}-e^{146}-e^{245}\right).$$

- $\mathfrak{cp}_3^m$: *For* $t \in \left(-\frac{1}{5m^2}, \infty\right)$,

$$\varphi(t) = \left(1+5m^2t\right)\left(e^{127}-e^{236}-e^{245}\right) + \left(1+5m^2t\right)^{\frac{6}{5}} \left(e^{347}+e^{567}\right) + \left(1+5m^2t\right)^{\frac{9}{10}} \left(e^{135}-e^{146}\right).$$

- $\mathfrak{cp}_4^m$: *For* $t \in \left(-\frac{5}{24m^2}, \infty\right)$,

$$\varphi(t) = \left(1+\frac{24}{5}m^2t\right)^{\frac{23}{24}} \left(e^{127}-e^{146}-e^{236}-e^{245}\right) + \left(1+\frac{24}{5}m^2t\right)^{\frac{29}{24}} \left(e^{347}+e^{567}\right) + \left(1+\frac{24}{5}m^2t\right)^{\frac{7}{8}} e^{135}.$$

- $\mathfrak{cp}_5^m$: *For* $t \in \left(-\frac{1}{5m^2}, \infty\right)$,

$$\varphi(t) = \left(1+5m^2t\right)\left(e^{127}-e^{146}-e^{245}\right) + \left(1+5m^2t\right)^{\frac{6}{5}} \left(e^{347}+e^{567}\right) + \left(1+5m^2t\right)^{\frac{9}{10}} \left(e^{135}-e^{236}\right).$$

- $\mathfrak{cp}_6^m$: *For* $t \in \left(-\frac{14}{3m^2}, \infty\right)$,

$$\varphi(t) = \left(1+\frac{3}{14}m^2t\right)^{\frac{13}{14}} \left(e^{127}+e^{135}-e^{146}-e^{236}-e^{245}\right) + \left(1+\frac{3}{14}m^2t\right)^{\frac{17}{14}} \left(e^{347}+e^{567}\right).$$

- $\mathfrak{cp}_7^m$: *For* $t \in \left(-\frac{5}{24m^2},\ \infty\right)$,

$$\varphi(t) = \left(1 + \frac{24}{5}m^2 t\right)^{\frac{9}{8}} \left(e^{127} + e^{347} + e^{567} - e^{236}\right) + \left(1 + \frac{24}{5}m^2 t\right)^{\frac{7}{8}} \left(e^{135} - e^{146} - e^{245}\right).$$

Remark 4.3. Direct computations show that, for each Lie algebra $\mathfrak{cp}_s^m$, the Ricci tensors of the metrics $\widetilde{g}_t$ induced by the solutions $\widetilde{\varphi}(t)$ to the coflow given in Corollary 4.2 are given by

$$\mathrm{Ric}(\widetilde{g}_t) = \left(\frac{1 - \alpha\, m^2 t}{1 + \gamma\, m^2 t}\right) \mathrm{Ric}(g_t),$$

where $\mathrm{Ric}(g_t)$ are the Ricci tensors of the metrics g_t induced by the solutions $\varphi(t)$ to the flow given in Remark 3.3. Thus, as in the flow case, only the solutions on S_1 are Einstein.

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Appendix

See Tables 4 and 5.

Table 4. Non-vanishing coefficients of the curvature of the metric g_t induced by the solutions of the LCP flow expresssed in the adapted basis $\{x_i\}_{i=1}^{7}$

Solvmanifold	$R(g_t)$
S_2	$R_{1367}(t)=R_{1637}(t)=\frac{-2}{3}C_2,\quad R_{1717}(t)=\frac{-16}{3}C_2,\quad R_{1736}(t)=\frac{4}{3}C_2,$ $R_{1212}(t)=R_{1414}(t)=R_{1515}(t)=-4\,C_2,$ $R_{1313}(t)=R_{1616}(t)=R_{3636}(t)=\frac{-7}{3}C_2,\quad R_{3737}(t)=R_{6767}(t)=\frac{-4}{3}C_2,$ $R_{2323}(t)=R_{2626}(t)=R_{3434}(t)=R_{3535}(t)=R_{4646}(t)=R_{5656}(t)=-2\,C_2,$ $R_{2424}(t)=R_{2525}(t)=R_{2727}(t)=R_{4545}(t)=R_{4747}(t)=R_{5757}(t)=-3\,C_2$
S_3	$R_{1212}(t)=\frac{-3}{2}C_3,\quad R_{1313}(t)=R_{1414}(t)=R_{1515}(t)=R_{1616}(t)=\frac{-17}{4}C_3,$ $R_{1367}(t)=R_{1457}(t)=-R_{1547}(t)=-R_{1637}(t)=-R_{3716}(t)=-R_{4715}(t)=$ $R_{5714}(t)=R_{6713}(t)=\frac{-3}{4}C_3,\quad R_{3645}(t)=R_{4536}(t)=\frac{-C_3}{2},$ $R_{1717}(t)=-9\,C_3,\quad R_{1736}(t)=R_{1745}(t)=R_{3617}(t)=R_{4517}(t)=\frac{3}{2}C_3,$ $R_{2323}(t)=R_{2424}(t)=R_{2525}(t)=R_{2626}(t)=R_{3636}(t)=R_{4545}(t)=-3\,C_3,$ $R_{3456}(t)=-R_{3546}(t)=-R_{4635}(t)=R_{5634}(t)=\frac{C_3}{4},\quad R_{2727}(t)=-4\,C_3,$ $R_{3434}(t)=R_{3535}(t)=R_{3737}(t)=R_{4646}(t)=R_{4747}(t)=R_{5656}(t)=$ $R_{5757}(t)=R_{6767}(t)=\frac{-9}{4}C_3$
S_4	$R_{1234}(t)=R_{1256}(t)=-R_{1423}(t)=-R_{1625}(t)=-R_{2314}(t)=-R_{2516}(t)=$ $R_{3412}(t)=R_{3456}(t)=-R_{3546}(t)=-R_{4635}(t)=R_{5612}(t)=R_{5634}(t)=\frac{C_4}{5},$ $R_{1367}(t)=-R_{1547}(t)=-R_{2467}(t)=R_{2647}(t)=-R_{4715}(t)=R_{4726}(t)=$ $R_{6713}(t)=-R_{6724}(t)=\frac{-3}{5}C_4,\quad R_{1313}(t)=R_{1515}(t)=\frac{-27}{5}C_4,$ $R_{1414}(t)=R_{1616}(t)=-4\,C_4,\quad R_{1457}(t)=-R_{1637}(t)=\frac{-4}{5}C_4,$ $R_{1736}(t)=R_{1745}(t)=R_{3617}(t)=R_{4517}(t)=\frac{7}{5}C_4,\quad R_{1212}(t)=\frac{-42}{5}C_4,$ $R_{2323}(t)=R_{2525}(t)=\frac{-24}{5}C_4,\quad R_{2424}(t)=R_{2626}(t)=\frac{-17}{5}C_4,$ $R_{2746}(t)=R_{4627}(t)=\frac{-6}{5}C_4,\quad R_{3434}(t)=R_{4646}(t)=R_{5656}(t)=\frac{-12}{5}C_4,$ $R_{3535}(t)=R_{3737}(t)=R_{5757}(t)=\frac{-16}{5}C_4,\quad R_{3636}(t)=R_{4545}(t)=-3\,C_4,$ $R_{3645}(t)=R_{4536}(t)=\frac{-2}{5}C_4,\quad R_{3716}(t)=-R_{5714}(t)=\frac{4}{5}C_4,$ $R_{4747}(t)=R_{6767}(t)=\frac{-9}{5}C_4,\quad R_{1717}(t)=\frac{-49}{5}C_4$

Table 5. Continuation of Table 4

	$R(g_t)$
S_5	$R_{1212}(t)=R_{1717}(t)=R_{2727}(t)=\frac{-25}{5}\,C_3,\quad R_{1313}(t)=R_{2323}(t)=-5\,C_3,$ $R_{1256}(t)=-R_{1625}(t)=-R_{2516}(t)=R_{5612}(t)=\frac{C_3}{4},$ $R_{1414}(t)=R_{2424}(t)=R_{4545}(t)=R_{4646}(t)=R_{5656}(t)=R_{5757}(t)=$ $R_{6767}(t)=\frac{-9}{4}\,C_3,\quad R_{1515}(t)=R_{2626}(t)=\frac{-7}{2}\,C_3,$ $R_{1457}(t)=-R_{2467}(t)=R_{5714}(t)=-R_{6724}(t)=\frac{-3}{4}\,C_3,\quad R_{3434}=-2\,C_3,$ $R_{1547}(t)=-R_{2647}(t)=R_{4715}(t)=-R_{4726}(t)=\frac{C_3}{2},\quad R_{3737}=-4\,C_3,$ $R_{1745}(t)=-R_{2746}(t)=R_{4517}(t)=-R_{4627}(t)=\frac{5}{4}\,C_3,\quad R_{4747}=-C_3,$ $R_{1616}(t)=R_{2525}(t)=\frac{-15}{4}\,C_3,\quad R_{3535}(t)=R_{3636}(t)=-3\,C_3$
S_6	$R_{1234}(t)=R_{1256}(t)=R_{3412}(t)=R_{3456}(t)=R_{5612}(t)=R_{5634}(t)=\frac{C_2}{6},$ $R_{1313}(t)=R_{1414}(t)=R_{1515}(t)=R_{1616}(t)=R_{2323}(t)=R_{2424}(t)=R_{2525}(t)=$ $R_{2626}(t)=\frac{-31}{12}\,C_2,\quad R_{1212}(t)=R_{1717}(t)=R_{2727}(t)=\frac{-16}{3}\,C_2,$ $R_{1324}(t)=-R_{1423}(t)=R_{1526}(t)=-R_{1625}(t)=-R_{2314}(t)=R_{2413}(t)=R_{2615}(t)=$ $R_{3546}(t)=-R_{3645}(t)=-R_{4536}(t)=R_{4635}(t)=\frac{C_2}{12},$ $R_{1367}(t)=R_{1457}(t)=-R_{1547}(t)=-R_{1637}(t)=R_{2357}(t)=-R_{2467}(t)=-R_{2537}(t)=$ $R_{2647}(t)=-R_{3716}(t)=-R_{3725}(t)=-R_{4715}(t)=R_{4726}(t)=R_{5714}(t)=R_{5723}(t)=$ $R_{6713}(t)=-R_{6724}(t)=\frac{-C_2}{3},$ $R_{1736}(t)=R_{1745}(t)=R_{2735}(t)=-R_{2746}(t)=R_{3527}(t)=R_{3617}(t)=R_{4517}(t)=$ $-R_{4627}(t)=\frac{2}{3}\,C_2,\quad R_{3535}(t)=R_{3636}(t)=R_{4545}(t)=R_{4646}(t)=\frac{-19}{12}\,C_2,$ $R_{3434}(t)=R_{3737}(t)=R_{4747}(t)=R_{5656}(t)=R_{5757}(t)=R_{6767}(t)=\frac{-4}{3}\,C_2,$
S_7	$R_{1234}(t)=R_{1256}(t)=-R_{1423}(t)=R_{1526}(t)=-R_{2314}(t)=R_{2615}(t)=R_{3412}(t)=$ $R_{3456}(t)=-R_{3645}(t)=-R_{4536}(t)=R_{5612}(t)=R_{5634}(t)=\frac{-18}{5}\,C_4,$ $R_{1313}(t)=R_{1616}(t)=R_{2424}(t)=R_{2525}(t)=R_{3535}(t)=R_{4646}(t)=\frac{-17}{5}\,C_4,$ $R_{1367}(t)=-R_{1637}(t)=-R_{2467}(t)=-R_{2537}(t)=R_{2735}(t)=-R_{2746}(t)=R_{3527}(t)=$ $-R_{3716}(t)=-R_{3725}(t)=-R_{4627}(t)=R_{6713}(t)=-R_{6724}(t)=\frac{-3}{5}\,C_4,$ $R_{1414}(t)=R_{1515}(t)=R_{1717}(t)=R_{4545}(t)=R_{4747}(t)=R_{5757}(t)=\frac{-36}{5}\,C_4,$ $R_{1736}(t)=R_{2357}(t)=R_{2647}(t)=R_{3617}(t)=R_{4726}(t)=R_{5723}(t)=\frac{6}{5}\,C_4,$ $R_{2323}(t)=R_{2626}(t)=R_{3636}(t)=\frac{-12}{5}\,C_4,\quad R_{2727}(t)=R_{3737}(t)=R_{6767}(t)=\frac{-9}{5}\,C_4$

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